Norms on the cohomology of hyperbolic 3-manifolds

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Abstract. We study the relationship between two norms on the first cohomology of a hyperbolic 3-manifold: the purely topological Thurston norm and the more geometric harmonic norm. Refining recent results of Bergeron, Şengün, and Venkatesh, we show that these norms are roughly proportional with explicit constants depending only on the volume and injectivity radius of the hyperbolic 3-manifold itself. Moreover, we give families of examples showing that some (but not all) qualitative aspects of our estimates are sharp. Finally, we exhibit closed hyperbolic 3-manifolds where the Thurston norm grows exponentially in terms of the volume and yet there is a uniform lower bound on the injectivity radius.

1 Introduction

Suppose M is a closed oriented hyperbolic 3-manifold. Our main goal here is to understand the relationship between two norms on $H^1(M;\mathbb{R})$: the purely topological Thurston norm and the more geometric harmonic norm. Precise definitions of these norms are given in Section 2, but for now here is an informal sketch.

The *Thurston norm* $\|\phi\|_{Th}$ of an integral class $\phi \in H^1(M;\mathbb{R})$ measures the topological complexity of the simplest surface dual to ϕ; it extends to all of $H^1(M;\mathbb{R})$ where its unit ball is a finite-sided polytope with rational vertices. It makes sense for any 3-manifold, though unlike in the hyperbolic case where it is nondegenerate, there can be nontrivial ϕ of norm 0. While it was introduced by Thurston in the 1970s [Thu2], its roots go back to the early days of topology, to questions about the genus of knots in the 3-sphere, and it has been extensively studied in many contexts.

Turning to geometry, as with any Riemannian manifold, the hyperbolic metric on M gives a norm on $H^1(M;\mathbb{R})$ which appears in the proof of the Hodge theorem. Specifically, if we identify $H^1(M;\mathbb{R})$ with the space of harmonic 1-forms, then the *harmonic norm* $\|\cdot\|_{L^2}$ is the one associated with the usual inner product on the level of forms:

$$\langle \alpha, \beta \rangle = \int_M \alpha \wedge *\beta$$

As it comes from a positive-definite inner product, the unit ball of $\|\cdot\|_{L^2}$ is a nice smooth ellipsoid. The harmonic norm appears, for example, in the Cheeger-Müller formula for the Ray-Singer analytic torsion of M [Che, Mül].

By Mostow rigidity, the hyperbolic metric on M is unique, and so a posteriori the harmonic norm depends solely on the underlying topology of M. While it is thus very natural to ask whether these two norms are related, this question was first considered in the very recent paper of Bergeron, Şengün, and Venkatesh [BŞV]. There, motivated by questions about torsion growth in the homology of arithmetic groups, they proved the following beautiful result:

1.1 Theorem [BŞV, 4.1]. *Suppose M_0 is a closed orientable hyperbolic 3-manifold. There exist constants C_1 and C_2, depending on M_0, so that for every finite cover M of M_0 one has*

$$\frac{C_1}{\mathrm{vol}(M)} \|\cdot\|_{Th} \leq \|\cdot\|_{L^2} \leq C_2 \|\cdot\|_{Th} \quad \text{on } H^1(M;\mathbb{R}). \tag{1.2}$$

In fact, their proof immediately gives that the constants C_1 and C_2 depend only on a lower bound on the injectivity radius $\mathrm{inj}(M_0)$, which is half the length of the shortest closed geodesic. Our main result is the following refinement of Theorem 1.1:

1.3 Theorem. *For all closed orientable hyperbolic 3-manifolds M one has*

$$\frac{\pi}{\sqrt{\mathrm{vol}(M)}} \|\cdot\|_{Th} \leq \|\cdot\|_{L^2} \leq \frac{10\pi}{\sqrt{\mathrm{inj}(M)}} \|\cdot\|_{Th} \quad \text{on } H^1(M;\mathbb{R}). \tag{1.4}$$

We also give families of examples which show that some (but not all) qualitative aspects of Theorem 1.3 are sharp. The first proves that the basic form of the first inequality in (1.4) cannot be improved:

1.5 Theorem. *There exists a sequence of M_n and $\phi_n \in H^1(M_n;\mathbb{R})$ so that*

(a) The quantities $\mathrm{vol}(M_n)$ and $\mathrm{inj}(M_n)$ both go to infinity as n does.

(b) The ratio $\dfrac{\|\phi_n\|_{Th}}{\|\phi_n\|_{L^2}\sqrt{\mathrm{vol}(M_n)}}$ is constant.

The next result concerns the second inequality of (1.4), and shows that the harmonic norm can blow up relative to the Thurston norm when the injectivity radius gets small.

1.6 Theorem. *There exists a sequence of M_n and $\phi_n \in H^1(M_n;\mathbb{R})$ so that*

(a) *The volumes of the M_n are uniformly bounded and $\mathrm{inj}(M_n) \to 0$ as $n \to \infty$.*

(b) $\lVert\phi_n\rVert_{L^2} / \lVert\phi_n\rVert_{Th} \to \infty$ *like* $\sqrt{-\log\left(\mathrm{inj}(M_n)\right)}$ *as* $n \to \infty$.

The growth of $\lVert\phi_n\rVert_{L^2} / \lVert\phi_n\rVert_{Th}$ in Theorem 1.6 is much slower than the most extreme behavior permitted by the second inequality in (1.4). We suspect that the examples in Theorem 1.6 have the worst possible behavior, but we are unable to improve (1.4) in that direction and believe doing so requires an entirely new approach.

1.7 Manifolds with large norms. A very intriguing conjecture of [BŞV] is that for congruence covers of a fixed *arithmetic* hyperbolic 3-manifold M_0, the size of the Thurston norm grows more slowly than you would naively expect. Specifically, any cover M of M_0 should have a nontrivial $\phi \in H^1(M;\mathbb{Z})$ where $\lVert\phi\rVert_{Th}$ is bounded by a polynomial in $\mathrm{vol}(M)$; in contrast, the usual estimates using a natural triangulation of M give only that there is a ϕ with $\lVert\phi\rVert_{Th}$ is at most exponential in $\mathrm{vol}(M)$. Our other contribution here is to give examples of closed hyperbolic 3-manifolds where the Thurston norm is exponentially large and yet there is a uniform lower bound on the injectivity radius.

1.8 Theorem. *There exist constants $C_1, C_2, \epsilon_1 > 0$ and a sequence of closed hyperbolic 3-manifolds M_n with $\mathrm{vol}(M_n) \to \infty$ where for all n:*

(a) $\mathrm{inj}(M_n) > \epsilon_1$,

(b) $b_1(M_n) = 1$,

(c) $\lVert\phi_n\rVert_{Th} > C_1 e^{C_2 \mathrm{vol}(M_n)}$ *where ϕ_n is a generator of $H^1(M_n;\mathbb{Z})$.*

Presumably, though we do not show this, the examples in Theorem 1.8 are nonarithmetic, and so point toward a divergence in behavior of the regulator term in the analytic torsion in the arithmetic and nonarithmetic cases, which is consistent with e.g. the experiments of [BD, §4].

1.9 Proof highlights. The arguments for the two inequalities of (1.4) in Theorem 1.3 are mostly independent. For the first inequality, we take a quite different approach from that of [BŞV]. Namely, to mediate between the topological Thurston norm and the analytical harmonic norm, we study what we call the *least-area norm*. For an integral class $\phi \in H^1(M;\mathbb{Z})$, this norm is simply the least area of any embedded smooth surface dual to ϕ. Using results on minimal surfaces of Schoen and Uhlenbeck, we show this new norm is uniformly comparable to the Thurston norm in the context of hyperbolic 3-manifolds (Theorem 3.4). Also, an argument using the coarea formula in geometric measure theory allows us to reinterpret the least-area norm as the L^1-analog of the L^2-based harmonic norm (Lemma 3.1). With these connections in place, the first inequality of (1.4) boils down to simply the Cauchy-Schwarz inequality.

In contrast, our proof of the second inequality in (1.4) follows the approach of [BŞV] closely. The key improvement is a refined upper bound on the L^∞-norm of a harmonic 1-form in terms of its harmonic norm. The latter result is Theorem 4.1 and is proved by a detailed analysis of the natural series expansion of a harmonic 1-form about a point in $\mathbb{H}^3$.

For the examples, Theorem 1.5 uses a simple construction involving finite covers, and Theorem 1.6 is based on Dehn filling a suitable 2-cusped hyperbolic 3-manifold. Finally, the proof of Theorem 1.8 involves gluing together two acylindrical homology handlebodies by a large power of a pseudo-Anosov; as with our prior work on integer homology spheres with large injectivity radius [BD], the key is controlling the homology of the resulting manifolds.

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2 The harmonic and Thurston norms

2.1 Conventions. Throughout this paper, all manifolds of any dimension are orientable and moreover oriented. All cohomology will have $\mathbb{R}$ coefficients unless noted otherwise.

2.2 The harmonic norm. For a closed Riemannian 3-manifold M, the natural positive-definite inner product on the space $\Omega^k(M)$ of real-valued k-forms is given by

$$\langle \alpha, \beta \rangle = \int_M \alpha \wedge *\beta \quad \text{where } * \text{ is the Hodge star operator.}$$

The *harmonic representative* of a class $[\alpha] \in H^k(M)$ is the unique one that minimizes $\langle \alpha, \alpha \rangle$; equivalently, if $\Delta = dd^* + d^* d$ is the Hodge Laplacian, it is the representative where $\Delta\alpha = 0$. Thus the cohomology $H^k(M)$ inherits the above inner product via the identification of it with the subspace of harmonic forms. The corresponding norm on $H^k(M)$ is, by definition, the *harmonic norm* $\| \cdot \|_{L^2}$ discussed in the introduction. Equivalently, it is defined by

$$\|\phi\|_{L^2} = \inf\left\{ \|\alpha\|_{L^2} \;\middle|\; \alpha \in \Omega^k(M) \text{ represents } \phi \right\} \tag{2.3}$$

For a 1-form α on M, a useful geometric viewpoint on $\|\alpha\|_{L^2}$ is the following. For a point p in M, denote the operator norm of the linear functional $\alpha_p \colon T_pM \to \mathbb{R}$ by $|\alpha_p|$; equivalently $|\alpha_p| = \sqrt{*(\alpha_p \wedge *\alpha_p)}$ which is also just the length of α_p under the metric-induced isomorphism of $T^pM \to T_pM$. The harmonic norm of α is then simply the L^2-norm of the associated function $|\alpha| \colon M \to \mathbb{R}_{\geq 0}$ since

$$\|\alpha\|_{L^2} = \sqrt{\int_M \alpha \wedge *\alpha} = \sqrt{\int_M |\alpha|^2 \, dVol} \tag{2.4}$$

Analogously, we define the L^1- and L^∞-norms of the 1-form α as

$$\|\alpha\|_{L^1} = \int_M |\alpha| \, dVol \quad \text{and} \quad \|\alpha\|_{L^\infty} = \max_{p \in M} |\alpha_p|. \tag{2.5}$$

2.6 The Thurston norm. For a connected surface, define $\chi_-(S) = \max\big(-\chi(S), 0\big)$; extend this to all surfaces via $\chi_-\big(S \sqcup S'\big) = \chi_-(S) + \chi_-\big(S'\big)$. For a compact irreducible 3-manifold M, the Thurston norm of $\phi \in H^1(M;\mathbb{Z}) \cong H_2(M, \partial M; \mathbb{Z})$ is defined by

$$\|\phi\|_{Th} = \min\left\{ \chi_-(S) \;\middle|\; S \text{ is a properly embedded surface dual to } \phi \right\}$$

The Thurston norm extends by continuity to all of $H^1(M)$, and the resulting unit ball is a finite-sided rational polytope [Thu2]. When M is hyperbolic, the Thurston norm is non-degenerate with $\|\phi\|_{Th} > 0$ for all nonzero ϕ; in general, it is only a seminorm.

3 The least area norm

To mediate between the harmonic and Thurston norms, we introduce two additional norms on the first cohomology of a closed Riemannian 3-manifold M, namely the least-area norm and the L^1-norm. In fact, these two norms coincide, but the differing perspectives they offer are a key tool used to prove the lower bound in Theorem 1.3.

For ϕ in $H^1(M;\mathbb{Z})$, let $\mathscr{F}_\phi$ be the collection of smooth maps $f\colon S \to M$ where S is a closed oriented surface with $f_*([S])$ dual to ϕ. The *least area norm* of ϕ is

$$\|\phi\|_{LA} = \inf\{\mathrm{Area}\big(f(S)\big) \mid f \in \mathscr{F}_\phi\}$$

By standard results in geometric measure theory, the value $\|\phi\|_{LA}$ is always realized by a smooth *embedded* surface $S \subset M$, whose components may be weighted by integer multiplicities; see e.g. [Has1, Theorem 1] for details. We will show below that $\|\cdot\|_{LA}$ is a seminorm on $H^1(M;\mathbb{Z})$ which extends continuously to a seminorm on all of $H^1(M)$.

In analogy with (2.3), we use the L^1-norm on 1-forms given in (2.5) to define the following function on $H^1(M)$:

$$\|\phi\|_{L^1} = \inf\{\|\alpha\|_{L^1} \mid \alpha \in \Omega^1(M) \text{ represents } \phi\}$$

Unlike the harmonic norm, the value $\|\phi\|_{L^1}$ is typically not realized by any a smooth form α. Despite this, it is easy to show that $\|\cdot\|_{L^1}$ is a seminorm on $H^1(M)$. As promised, these two new norms are in fact the same:

3.1 Lemma. *$\|\phi\|_{LA} = \|\phi\|_{L^1}$ for all $\phi \in H^1(M;\mathbb{Z})$.*

Note that one consequence of Lemma 3.1 is the promised fact that $\|\cdot\|_{LA}$ extends continuously from $H^1(M;\mathbb{Z})$ to a seminorm on all of $H^1(M)$.

Proof. To show $\|\phi\|_{LA} \geq \|\phi\|_{L^1}$, let S be a smooth embedded surface dual to ϕ of area $\|\phi\|_{LA}$. For each $\epsilon > 0$, consider a dual 1-form α_ϵ which is supported in an ϵ-neighborhood of S and is a slight smoothing of $\frac{1}{2\epsilon} d\big(\text{signed distance to S}\big)$ there. An easy calculation in Fermi coordinates shows that $\|\alpha_\epsilon\|_{L^1} \to \mathrm{Area}(S) = \|\phi\|_{LA}$ as $\epsilon \to 0$, giving $\|\phi\|_{LA} \geq \|\phi\|_{L^1}$.

To establish $\|\phi\|_{LA} \leq \|\phi\|_{L^1}$, let α be any representative of ϕ. Since ϕ is an integral class, by integrating α we get a smooth map $f\colon M \to S^1$ so that $\alpha = f^*(dt)$, where we have parameterized $S^1 = \mathbb{R}/\mathbb{Z}$ by $t \in [0,1]$. For each $t \in [0,1]$, the set $S_t = f^{-1}(t)$ is a smooth surface for almost all t. For all t, we define $\mathrm{Area}(S_t)$ to be the 2-dimensional Hausdorff measure of S_t. As the operator norm $|\alpha|$ is equal to the 1-Jacobian of the map f, the Coarea Formula [Mor, Theorem 3.8] is precisely

$$\int_M |\alpha| \, dVol = \int_0^1 \mathrm{Area}(S_t)\, dt \tag{3.2}$$

Since a function on $[0,1]$ is less than or equal to its mean on a set of positive measure, there are many t so that S_t is smooth and $\mathrm{Area}(S_t) \leq \|\alpha\|_{L^1}$. Taking the infimum over representatives α of ϕ gives $\|\phi\|_{LA} \leq \|\phi\|_{L^1}$ as desired. □

3.3 Relationship with the Thurston norm. When M is hyperbolic, the least area norm is very closely related to the Thurston norm:

3.4 Theorem. *For any closed hyperbolic 3-manifold M and $\phi \in H^1(M)$ one has:*

$$\pi \|\phi\|_{Th} \leq \|\phi\|_{LA} \leq 2\pi \|\phi\|_{Th} \tag{3.5}$$

The moral behind this result is that *stable* minimal surfaces in hyperbolic 3-manifolds have uniformly bounded intrinsic curvature [Sch], and hence area and genus are essentially proportionate. Specifically, we will use the following fact, which was first observed by Uhlenbeck [Uhl] in unpublished work.

3.6 Lemma. *For any stable closed minimal surface S in a hyperbolic 3-manifold:*

$$\pi \chi_-(S) \leq \mathrm{Area}(S) \leq 2\pi \chi_-(S). \tag{3.7}$$

Proof of Lemma 3.6. The proof is essentially the same as [Has2, Lemma 6], which you should see for details. As S is minimal, its intrinsic curvature $K \colon S \to \mathbb{R}$ is bounded above by that of hyperbolic space, i.e. by -1. In particular, by Gauss-Bonnet, every component of S has negative Euler characteristic and moreover

$$2\pi \chi_-(S) = -2\pi \chi(S) = \int_S -K \, dA \geq \int_S 1 \, dA = \mathrm{Area}(S)$$

giving the righthand inequality in (3.7). For the other inequality, since S is stable, the main argument in [Has2, Lemma 6] with the test function $f = 1$ gives that $\pi \chi_-(S) \leq \mathrm{Area}(S)$ as desired. □

Proof of Theorem 3.4. Pick a surface S dual to ϕ which is incompressible and realizes the Thurston norm, i.e. $\|\phi\|_{Th} = \chi_-(S)$. Since S is incompressible, by [FHS] we can assume that S has least area in its isotopy class and hence is a stable minimal surface. Thus by Lemma 3.6 we have

$$\|\phi\|_{LA} \leq \mathrm{Area}(S) \leq 2\pi \chi_-(S) = 2\pi \|\phi\|_{Th}.$$

proving the second half of (3.5).

For the other inequality, suppose S is a least area surface dual to ϕ. Note again that S must be a stable minimal surface, and so Lemma 3.6 gives

$$\pi \|\phi\|_{Th} \leq \pi \chi_-(S) \leq \mathrm{Area}(S) = \|\phi\|_{LA}$$

proving the rest of (3.5). □

4 Pointwise bounds on harmonic 1-forms

Let M be a closed hyperbolic 3-manifold. A key tool used by [BŞV] in their proof of both inequalities (1.2) in Theorem 1.1 is that there is a constant C, depending somehow on the injectivity radius of the hyperbolic 3-manifold, so that

$$\| \cdot \|_{L^\infty} \leq C \| \cdot \|_{L^2}$$

In our parallel Theorem 1.3, we will use this fact only in the second inequality in (1.4), after first refining it into:

4.1 Theorem. *If α is a harmonic 1-form on a closed hyperbolic manifold M then*

$$\|\alpha\|_{L^\infty} \leq \frac{5}{\sqrt{\operatorname{inj}(M)}} \|\alpha\|_{L^2} \tag{4.2}$$

4.3 Remark. While the 5 in (4.2) can be improved, it seems unlikely that the exponent on $\operatorname{inj}(M)$ can be significantly reduced. If T_ϵ is a tube of volume 1 around a core geodesic of length 2ϵ, then there is a harmonic 1-form α_ϵ on T_ϵ (namely dz in cylindrical coordinates) so that

$$\frac{\|\alpha_\epsilon\|_{L^\infty}}{\|\alpha_\epsilon\|_{L^2}} \asymp \left(\epsilon \log(\epsilon^{-1})\right)^{-1/2}$$

Our proof of Theorem 4.1 starts by understanding how certain harmonic 1-forms behave on balls in $\mathbb{H}^3$ via

4.4 Lemma. *If $f\colon \mathbb{H}^3 \to \mathbb{R}$ is harmonic and B is a ball of radius r centered about p then*

$$\left| df_p \right| \leq \frac{1}{\sqrt{v(r)}} \|df\|_{L^2(B)} \tag{4.5}$$

where

$$v(r) = 6\pi \left(r + 2r \operatorname{csch}^2(r) - \coth(r)\left(r^2 \operatorname{csch}^2(r) + 1\right)\right) \tag{4.6}$$

The function $v\colon \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is a montone increasing bijection with $v(r) \sim 4\pi r^3/3$ as $r \to 0$ and $v(r) \sim 6\pi r$ as $r \to \infty$. The estimate in (4.5) is sharp; in fact, our proof gives a single harmonic function f for which (4.5) is an equality for all r. Theorem 4.1 will follow directly from Lemma 4.4 when r is large, but $v(r)$ goes to 0 too fast as $r \to 0$ to immediately give (4.2) when r is small.

The missing ingredient needed to prove Theorem 4.1 is the following notion. A *Margulis number* for $M = \Gamma\backslash\mathbb{H}^3$ with $\Gamma \leq \operatorname{Isom}^+(\mathbb{H}^3)$ is a $\mu > 0$ so that for all $p \in \mathbb{H}^3$ the subgroup

$$\left\langle \gamma \in \Gamma \;\middle|\; d\big(p, \gamma(p)\big) < \mu \right\rangle$$

is abelian. For example, $\mu = 0.1$ is a Margulis number for any such M [Mey, Theorem 2], and here we will use that $\mu = 0.29$ is a Margulis number whenever $H^1(M) \neq 0$ by [CS]. For any fixed Margulis number μ, define

$$M_{thick} = \{m \in M \mid \mathrm{inj}_m M \geq \mu/2\} \quad \text{and} \quad M_{thin} = \{m \in M \mid \mathrm{inj}_m M < \mu/2\}$$

When M is closed, the thin part M_{thin} is a disjoint union of tubes about the finitely many closed geodesics of length less than μ. We will need:

4.7 Lemma. *Suppose M is a hyperbolic 3-manifold with Margulis number μ, and set $\epsilon = \min\{\mathrm{inj}(M), \mu/2\}$. Let $\pi\colon \mathbb{H}^3 \to M$ be the universal covering map. If $B \subset \mathbb{H}^3$ is a ball of radius $\mu/2$, then*

$$\max_{m \in M} \left| B \cap \pi^{-1}(m) \right| \leq \frac{\mu}{\epsilon} \tag{4.8}$$

We first assemble the pieces and prove Theorem 4.1 assuming Lemmas 4.4 and 4.7.

Proof of Theorem 4.1. We will assume that $H^1(M) \neq 0$ as otherwise the only harmonic 1-form is identically zero. Since $H^1(M) \neq 0$, Theorem 1.1 of [CS] gives that 0.29 is a Margulis number for M. Setting $\mu = 0.29$ and $\epsilon = \mathrm{inj}(M)$, there are two cases depending on how ϵ compares to $\mu/2$.

First we do the easy case of when $\epsilon \geq \mu/2$. Let $m \in M$ be a point where $|\alpha_m|$ is maximal. Take $\pi\colon \mathbb{H}^3 \to M$ to be the universal covering map and set $\widetilde{\alpha} = \pi^*(\alpha)$. Fix a ball $B \subset \mathbb{H}^3$ of radius ϵ centered at a point p in $\pi^{-1}(m)$. As α is harmonic on the compact manifold M, it is both closed and coclosed; as these are local properties, the same is true for $\widetilde{\alpha}$. As $\mathbb{H}^3$ is contractible and $\widetilde{\alpha}$ is closed, we have $\widetilde{\alpha} = df$ for some $f\colon \mathbb{H}^3 \to \mathbb{R}$. Moreover f is harmonic since $\Delta f = (d^* \circ d) f = d^* \widetilde{\alpha} = 0$. Using Lemma 4.4 we get

$$\|\alpha\|_{L^\infty} = |\alpha_m| = \left|\widetilde{\alpha}_p\right| \leq \frac{1}{\sqrt{\nu(\epsilon)}} \|\widetilde{\alpha}|_B\|_{L^2} \leq \frac{1}{\sqrt{\nu(\epsilon)}} \|\alpha\|_{L^2}$$

where the last inequality follows as $\pi|_B$ is injective. The inequality (4.2) now follows from the fact that $\sqrt{\epsilon/\nu(\epsilon)} < 3.5 < 5$ for $\epsilon \geq \mu/2 = 0.145$.

Now suppose $\epsilon < \mu/2$. We take the same setup as before except that $B \subset \mathbb{H}^3$ will now have radius $\mu/2$. By Lemma 4.7 it follows that

$$\|\widetilde{\alpha}|_B\|_{L^2} \leq \sqrt{\frac{\mu}{\epsilon}} \left\|\alpha|_{\pi(B)}\right\|_{L^2} \leq \sqrt{\frac{\mu}{\epsilon}} \|\alpha\|_{L^2}$$

Hence

$$\|\alpha\|_{L^\infty} = |\alpha_m| = \left|\widetilde{\alpha}_p\right| \leq \frac{1}{\sqrt{\nu(\mu/2)}} \|\widetilde{\alpha}|_B\|_{L^2} \leq \sqrt{\frac{\mu}{\nu(\mu/2)}} \frac{1}{\sqrt{\epsilon}} \|\alpha\|_{L^2}$$

As $\sqrt{\mu/\nu(\mu/2)} \approx 4.78$ at $\mu = 0.29$, we have proved 4.2 in this case as well. □

Turning to the lemmas, we start with the easier one which follows from a simple geometric argument.

Proof of Lemma 4.7. We can assume $\epsilon = \mathrm{inj}(M) < \mu/2$ as otherwise $\pi|_B$ is injective and the result is immediate since the righthand side of (4.8) is 2. The basic idea of the proof is that the worst-case senario is when m is on a closed geodesic C of minimal length 2ϵ, and B is centered at a point of $\pi^{-1}(C)$. Then B can contain $n+1$ points in $\pi^{-1}(C)$ where $n = \lfloor \mu/2\epsilon \rfloor$; using that $n+1 \leq \mu/\epsilon$ then gives (4.8). We now give the detailed proof.

If $m \in M_{thick}$, then any pair of distinct points in $\pi^{-1}(m)$ are distance at least μ apart, and hence at most one is in the open ball B; as $\mu/\epsilon \geq 2$, we have proven (4.8) in this case.

If $m \in M_{thin}$, it lies in some tube T about a short closed geodesic C. The components of $\pi^{-1}(T)$ are radius R neighborhoods about the various geodesic lines in $\pi^{-1}(C)$. First, note that $B \cap \pi^{-1}(M)$ must lie in a single component $\widetilde{T}$ of $\pi^{-1}(T)$; let $\gamma \in \Gamma$ generate the stabilizer of $\widetilde{T}$. Pick a $\widetilde{m}_0 \in \widetilde{T} \cap \pi^{-1}(m)$; then $\pi^{-1}(m)$ consists of $\gamma^n \cdot \widetilde{m}_0$ for $n \in \mathbb{Z}$. Adjust $\widetilde{m}_0$ if necessary so that $\widetilde{m}_0 \in B$ and any $\widetilde{m}_n = \gamma^n \cdot \widetilde{m}_0$ in B has $n \geq 0$. Since $d(\widetilde{m}_0, \widetilde{m}_n) \geq n \cdot \mathrm{len}(C) \geq 2n\epsilon$, if $\widetilde{m}_n \in B$ we have $n \leq \mu/2\epsilon$. So there are at most $(\mu/2\epsilon)+1$ of the $\widetilde{m}_i$ in B. Since $2\epsilon \leq \mu$, we get $\left|B \cap \pi^{-1}(m)\right| \leq \mu/\epsilon$ as desired. □

While the calculations in the proof of Lemma 4.4 are somewhat involved (unsurprisingly given the formula for $\nu(r)$), the basic idea is simple and we sketch it now. Using the natural series expansion for harmonic functions about p, we show that after a suitable isometry of $\mathbb{H}^3$ fixing p we have

$$df = a\omega + \beta$$

where $a \in \mathbb{R}_{\geq 0}$, the 1-form ω is a fixed and independent of f with $\left|\omega_p\right| = 1$, the 1-form β vanishes at 0, and β is orthogonal to ω in $L^2\big(B_r(p)\big)$ for all r. Then $df_p = a\omega_p$ and for each such $B = B_r(p)$ the orthogonality of ω and β implies $a\|\omega\|_{L^2(B)} \leq \left\|df\right\|_{L^2(B)}$. Hence

$$\left|df_p\right| = \left|a\omega_p\right| = a \leq \frac{\left\|df\right\|_{L^2(B)}}{\|\omega\|_{L^2(B)}}$$

Thus we simply define $\nu(r)$ to be $\|\omega\|^2_{L^2(B_r(p))}$ and Lemma 4.4 will then follow by calculating $\nu(r)$ explicitly.

4.9 Series expansions of harmonic functions. We now describe in detail the key tool used to prove Lemma 4.4: that every harmonic function $f\colon \mathbb{H}^3 \to \mathbb{R}$ has a series expansion in terms of a certain basis $\{\Psi_{\ell m}\}$ of harmonic functions centered around

p; throughout, see [Min] or [EGM, §3.5] for details. Consider the spherical coordinates $(r,\phi,\theta) \in [0,\infty) \times [0,\pi) \times [0,2\pi)$ on $\mathbb{H}^3$ centered about p; in these coordinates, the metric is:

$$ds^2_{\mathbb{H}^3} = dr^2 + \sinh^2(r)\,ds^2_{S^2} = dr^2 + \sinh^2(r)\left(d\phi^2 + \sin^2(\phi)\,d\theta^2\right)$$

For $\ell \in \mathbb{Z}_{\geq 0}$, define $\psi_\ell \colon \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ by

$$\psi_\ell(r) = \frac{\Gamma\left(\frac{3}{2}\right)\Gamma(\ell+2)}{\Gamma\left(\ell+\frac{3}{2}\right)} \tanh^\ell\left(\frac{r}{2}\right) \cdot {}_2F_1\left(-\frac{1}{2}, \ell; \ell+\frac{3}{2}; \tanh^2\left(\frac{r}{2}\right)\right)$$

where ${}_2F_1$ is the usual hypergeometric function and Ψ_0 is simply the constant function 1. If $Y_{\ell m}(\phi,\theta)$ for $\ell \geq 0$ and $-\ell \leq m \leq \ell$ are the usual basis for the *real* spherical harmonics on S^2, define:

$$\Psi_{\ell m} = \psi_\ell(r)\,Y_{\ell m}(\phi,\theta)$$

These are harmonic functions on all of $\mathbb{H}^3$, and moreover every harmonic function $f \colon \mathbb{H}^3 \to \mathbb{R}$ has unique $a_{\ell m} \in \mathbb{R}$ so that

$$f = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} a_{\ell m} \Psi_{\ell m} \tag{4.10}$$

where the series converges absolutely and uniformly on compact subsets of $\mathbb{H}^3$. We will use the following elementary properties of these functions:

4.11 Lemma.

(a) *$\psi_\ell(r)$ vanishes to order exactly ℓ at $r = 0$. Consequently, $\Psi_{\ell m}$ vanishes to order exactly $\ell - 1$ at p.*

(b) *On any ball B about p, the functions $\Psi_{\ell m}$ are orthogonal in $L^2(B)$.*

(c) *The 1-forms $\omega_{\ell m} = d\Psi_{\ell m}$ are also orthogonal in each $\Omega^1(B)$.*

Proof. The claim (a) follows since $\tanh\frac{r}{2} = \frac{1}{2}r + O(r^2)$ for small r and in addition ${}_2F_1(a,b;c;0) = 1$. The second claim (b) is an easy consequence of the fact that the $Y_{\ell m}$ are orthonormal as elements of $L^2\left(S^2\right)$. For part (c), we have

$$\omega_{\ell m} = d\left(\psi_\ell Y_{\ell m}\right) = Y_{\ell m}\frac{\partial \psi_\ell}{\partial r}dr + \psi_\ell\, dY_{\ell m}$$

and then observing that the cross-terms vanish we get

$$\omega_{\ell m} \wedge *\omega_{kn} = Y_{\ell m}Y_{kn}\frac{\partial \psi_\ell}{\partial r}\frac{\partial \psi_k}{\partial r}dVol + \psi_\ell\psi_k\, dY_{\ell m} \wedge *dY_{kn}. \tag{4.12}$$

To compute $\langle \omega_{\ell m}, \omega_{kn} \rangle$, we integrate the above over B and show it vanishes unless $(\ell, m) = (k, n)$. In fact, we argue that the integral of (4.12) over each S^2 where r is fixed is 0. For the first term on the right-hand side of (4.12) this follows immediately from the orthogonality of the $Y_{\ell m}$ in $L^2(S^2)$. For the second term, note that $* dY_{kn} = (\overline{*} dY_{kn}) \wedge dr$ where $\overline{*}$ is the Hodge star operator on S^2, and hence the real claim is that $\langle dY_{\ell m}, dY_{kn} \rangle_{S^2}$ vanishes. This can be deduced from the orthogonality of $Y_{\ell m}$ and Y_{kn} via

$$\langle dY_{\ell m}, dY_{kn} \rangle_{S^2} = \left\langle Y_{\ell m}, d^{\overline{*}} dY_{kn} \right\rangle_{S^2} = \left\langle Y_{\ell m}, \Delta_{S^2} Y_{kn} \right\rangle_{S^2} = k(k+1) \langle Y_{\ell m}, Y_{kn} \rangle_{S^2}$$

where the last equality holds because Y_{kn} is an eigenfunction of Δ_{S^2}. This finishes the proof of (c). □

We now prove Lemma 4.4 using the approach sketched earlier.

Proof of Lemma 4.4. Let the $a_{\ell m}$ be the coefficients in the expansion (4.10) for f, and note we get a corresponding expansion

$$df = \sum_{\ell=1}^{\infty} \sum_{m=-\ell}^{\ell} a_{\ell m} \omega_{\ell m} \quad \text{where } \omega_{\ell m} = d\Psi_{\ell m}.$$

Here the series converges absolutely and uniformly on B, and henceforth we view B as the domain of all our 1-forms. Defining

$$\eta = a_{1,-1}\omega_{1,-1} + a_{1,0}\omega_{1,0} + a_{1,0}\omega_{1,0}$$

we see by Lemma 4.11(a) that $\eta_p = df_p$. Because of the orthogonality of the $\omega_{\ell m}$ on B, we know $\|\eta\|_{L^2} \leq \|df\|_{L^2}$ and so it suffices to prove (4.5) for η, or indeed for the components of η. In fact, because we can use an isometry of $\mathbb{H}^3$ fixing 0 to interchange the $Y_{1,m}$, it suffices to establish (4.5) for the single form $\omega_{1,0}$, or indeed any multiple of it. Thus Lemma 4.4 will follow immediately from the next result. □

4.13 Lemma. *For the harmonic 1-form $\omega = \sqrt{3\pi} \cdot \omega_{1,0}$ we have $|\omega_p| = 1$ and $\|\omega\|_{L^2(B_r(p))} = v(r)$.*

Proof. The ψ_ℓ actually have alternate expressions in terms of elementary functions; in the case of interest, using that

$${}_2F_1\left(1/2, 1; 3/2; x^2\right) = \sum_{n=0}^{\infty} \frac{(1/2)_n (1)_n}{(3/2)_n} \frac{x^{2n}}{n!} = \sum_{n=0}^{\infty} \frac{x^{2n}}{2n+1} = \frac{\operatorname{arctanh}(x^2)}{x}$$

and applying two contiguous relations for hypergeometric functions yields

$$ {}_2F_1\left(-1/2, 1; 5/2; x^2\right) = \frac{3\left(x^3 - \left(x^2-1\right)^2 \operatorname{arctanh}(x) + x\right)}{8x^3} $$

and hence

$$ \psi_1(r) = \coth(r) - r\operatorname{csch}^2(r) = \frac{\sinh(r)\cosh(r) - r}{\sinh^2(r)} = \frac{2}{3}r - \frac{4}{45}r^3 + O(r^5). $$

As $Y_{1,0}$ is $\sqrt{3/4\pi}\cos\phi$, writing

$$ \omega = \sqrt{3\pi}\cdot\omega_{1,0} = \frac{3}{2}d\left(\psi_1\cos\phi\right) $$

in terms of the orthonormal coframe

$$ \widehat{dr} = dr \qquad \widehat{d\phi} = \sinh(r)\,d\phi \qquad \widehat{d\theta} = \sinh(r)\sin(\phi)\,d\theta $$

gives

$$ \omega = \frac{3}{2}\left(\left(\partial_r\psi_1\right)\cos(\phi)\widehat{dr} - \frac{\psi_1\sin(\phi)}{\sinh(r)}\widehat{d\phi}\right) \quad \text{where } \partial_r\psi_1 = \frac{\partial\psi_1}{\partial r} = 2\cdot\frac{r\coth(r)-1}{\sinh^2(r)}, $$

and hence

$$ |\omega|^2 = \frac{9}{4}\left(\left(\partial_r\psi_1\right)^2\cos^2(\phi) + \frac{\psi_1^2\sin^2\phi}{\sinh^2(r)}\right) $$

Approaching the origin along the ray $\phi = 0$ gives:

$$ \left|\omega_p\right| = \frac{3}{2}\lim_{r\to 0}\frac{\partial\psi_1}{\partial r} = 3\lim_{r\to 0}\frac{r\coth(r)-1}{\sinh^2(r)} = 1 $$

Computing the L^2-norm of ω on B gives

$$ \begin{aligned} \|\omega\|_{L^2}^2 &= \int_B |\omega|^2 dVol = \int_0^R\int_0^\pi\int_0^{2\pi} |\omega|^2\sinh^2(r)\sin(\phi)\ d\theta\, d\phi\, dr \\ &= \frac{9\pi}{2}\int_0^R\int_0^\pi \left(\partial_r\psi_1\right)^2\sinh^2(r)\cos^2(\phi)\sin(\phi) + \psi_1^2\sin^3(\phi)\ d\phi\, dr \\ &= 3\pi\int_0^R \left(\partial_r\psi_1\right)^2\sinh^2(r) + 2\psi_1^2\ dr \qquad\qquad (4.14) \\ &= 6\pi\int_0^R \coth^2(r) + 2\operatorname{csch}^2(r) - 6r\coth(r)\operatorname{csch}^2(r) \\ &\qquad\qquad + r^2\operatorname{csch}^2(r)\left(2\coth^2(r) + \operatorname{csch}^2(r)\right)dr \\ &= 6\pi\left(R + 2R\operatorname{csch}^2(R) - \coth(R)\left(R^2\operatorname{csch}^2(R) + 1\right)\right) \end{aligned} $$

which proves the lemma. □

5 Proof of Theorem 1.3

This section is devoted to the proof of

1.3 Theorem. *For all closed orientable hyperbolic 3-manifolds M one has*

$$\frac{\pi}{\sqrt{\mathrm{vol}(M)}}\|\cdot\|_{Th} \le \|\cdot\|_{L^2} \le \frac{10\pi}{\sqrt{\mathrm{inj}(M)}}\|\cdot\|_{Th} \quad \text{on } H^1(M;\mathbb{R}). \tag{1.4}$$

Proof of Theorem 1.3. We start with the lower bound in (1.4), where we use the two guises of the least-area/L^1–norm to mediate between the Thurston and harmonic norms and thereby reduce the claim to the Cauchy-Schwarz inequality. Suppose $\phi \in H^1(M)$ and let α be the harmonic 1-form representing ϕ. By Theorem 3.4 and Lemma 3.1 we have

$$\pi\|\phi\|_{Th} \le \|\phi\|_{LA} = \|\phi\|_{L^1}$$

From the definition of the L^1-norm, we have $\|\phi\|_{L^1} \le \|\alpha\|_{L^1}$, and applying Cauchy-Schwarz to the pair $|\alpha|\colon M \to \mathbb{R}$ and the constant function 1 gives

$$\pi\|\phi\|_{Th} \le \|\alpha\|_{L^1} = \||\alpha|\cdot 1\|_{L^1} \le \|\alpha\|_{L^2}\|1\|_{L^2} = \|\alpha\|_{L^2}\sqrt{\mathrm{vol}(M)} \tag{5.1}$$

Since $\|\phi\|_{L^2} = \|\alpha\|_{L^2}$ by definition, dividing (5.1) through by $\sqrt{\mathrm{vol}(M)}$ gives the first part of (1.4).

The proof of the upper-bound in (1.4) is essentially the same as given in [BŞV] for the corresponding part of (1.2), but using the upgraded Theorem 4.1 to relate $\|\cdot\|_{L^\infty}$ and $\|\cdot\|_{L^2}$ and so give a sharper result. By continuity of the norms, it suffices to prove the upper bound for $\phi \in H^1(M;\mathbb{Z})$. Using Theorem 3.4, fix a surface S dual to ϕ of area at most $2\pi\|\phi\|_{Th}$; by definition, this means that for every *closed* 2-form β on M one has $\int_M \beta\wedge\alpha = \int_S \beta$. If α is the harmonic representative of ϕ, then $d^*\alpha = -*d*\alpha = 0$, and so it follows that $*\alpha$ is closed. Hence

$$\begin{aligned}\|\alpha\|_{L^2}^2 &= \int_M \alpha\wedge *\alpha = \int_M *\alpha\wedge\alpha = \int_S *\alpha \\ &\le \int_S |{*\alpha}|\, dA = \int_S |\alpha|\, dA \le \int_S \|\alpha\|_{L^\infty}\, dA \\ &\le \|\alpha\|_{L^\infty}\mathrm{Area}(S) \le 2\pi\|\alpha\|_{L^\infty}\|\phi\|_{Th}.\end{aligned} \tag{5.2}$$

Applying (4.2) from Theorem 4.1, we get

$$\|\alpha\|_{L^2}^2 \le 10\pi\epsilon^{-1/2}\|\alpha\|_{L^2}\|\phi\|_{Th}$$

Dividing through by $\|\alpha\|_{L^2}$ gives the upper bound in (1.4), proving the theorem. □

6 Families of examples

This section is devoted to proving Theorems 1.5 and 1.6, starting with the former as it is easier.

1.5 Theorem. *There exists a sequence of M_n and $\phi_n \in H^1(M_n;\mathbb{R})$ so that*

(a) The quantities $\mathrm{vol}(M_n)$ *and* $\mathrm{inj}(M_n)$ *both go to infinity as n does.*

(b) The ratio $\dfrac{\|\phi_n\|_{Th}}{\|\phi_n\|_{L^2}\sqrt{\mathrm{vol}(M_n)}}$ *is constant.*

Proof of Theorem 1.5. The examples M_n will be built from a tower of finite covers, where the ϕ_n are pullbacks of some fixed class $\phi_0 \in H^1(M_0)$. To analyze that situation, first consider a degree d covering map $\pi : \widetilde{M} \to M$ and some $\phi \in H^1(M)$; as we now explain, the norms of ϕ and $\pi^*(\phi)$ differ by factors depending only on d. If α is the harmonic representative of ϕ, then as being in the kernel of the Laplacian is a local property, the form $\pi^*(\alpha)$ must be the harmonic representative of $\pi^*(\phi)$. It follows that

$$\left\|\pi^*(\phi)\right\|_{L^2} = \sqrt{d}\cdot\left\|\phi\right\|_{L^2}.$$

In contrast, it is a deep theorem of Gabai [Gab, Cor. 6.18] that

$$\left\|\pi^*(\phi)\right\|_{Th} = d\cdot\left\|\phi\right\|_{Th}$$

Thus, the ratio

$$\frac{\|\cdot\|_{Th}}{\|\cdot\|_{L^2}\sqrt{\mathrm{vol}(\cdot)}}$$

is the same for both (M,ϕ) and $\left(\widetilde{M},\pi^*(\phi)\right)$.

To prove the theorem, let M_0 be any closed hyperbolic 3-manifold with a nonzero class $\phi_0 \in H^1(M_0)$ and choose a tower of finite covers

$$M_0 \leftarrow M_1 \leftarrow M_2 \leftarrow M_3 \leftarrow \cdots$$

where $\mathrm{inj}(M_n) \to \infty$. Taking $\phi_n \in H^1(M_n)$ to be the pullback of ϕ, we have constructed pairs (M_n,ϕ_n) which have all the claimed properties. □

6.1 Harmonic forms on tubes. To prove Theorem 1.6, we will need:

6.2 Lemma. *Suppose V is a tube in a closed hyperbolic 3-manifold M with a core C of length ϵ and depth R. If α is a harmonic 1-form on M with $\int_C \alpha = 1$ then*

$$\|\alpha|_V\|_{L^2} \geq \sqrt{\frac{2\pi}{\epsilon}\log(\cosh R)} \tag{6.3}$$

Before proving the lemma, we give coordinates on the tube V and do some preliminary calculations. Specifically, consider cylindrical coordinates $(r,\theta,z) \in [0,R] \times [0,2\pi] \times [0,\epsilon]$ with the additional identification $(r,\theta,\epsilon) \sim (r,\theta+\theta_0,0)$, where the twist angle θ_0 is determined by the geometry of the tube; the metric on V is then given by

$$g_{\mathbb{H}^3} = dr^2 + \sinh^2(r)\,d\theta^2 + \cosh^2(r)\,dz^2$$

Since $\{dr, \sinh(r)\,d\theta, \cosh(r)\,dz\}$ gives an orthonormal basis of 1-forms at each point of V, we have

$$\mathrm{Vol}(V) = \int_V dVol = \int_0^\epsilon \int_0^{2\pi} \int_0^R \sinh(r)\cosh(r)\,dr\,d\theta\,dz = \pi\epsilon\sinh^2(R) \tag{6.4}$$

Note that the form dz is compatible with the identification of $z=0$ with $z=\epsilon$, giving a 1-form on V. Now the form dz is closed and also coclosed since

$$d^*(dz) = -*d*dz = -*d\big(\tanh(r)\,dr\wedge d\theta\big) = -*0 = 0$$

and hence dz is harmonic. Notice $\omega = \frac{1}{\epsilon}dz$ is a reasonable candidate for $\alpha|_V$ in Lemma 6.2 as $\int_C \omega = 1$. For this form, we have

$$\begin{aligned}\|\omega\|_{L^2}^2 &= \int_V \omega\wedge *\omega = \frac{1}{\epsilon^2}\int_V dz\wedge\big(\tanh(r)\,dr\wedge d\theta\big) \\ &= \frac{1}{\epsilon^2}\int_0^\epsilon\int_0^{2\pi}\int_0^R \tanh(r)\,dr\,d\theta\,dz = \frac{2\pi}{\epsilon}\log(\cosh R)\end{aligned} \tag{6.5}$$

Thus Lemma 6.2 can be interpreted as saying that the harmonic norm of $\alpha|_V$ is at least that of this explicit ω, and we take this viewpoint in the proof itself.

Proof of Lemma 6.2. From now on, we denote $\alpha|_V$ by α and we use only that α is closed, coclosed, and $\int_C \alpha = 1$. There is an action of $G = S^1 \times S^1$ on V by isometries, namely translations in the θ and z coordinates. First, we show it suffices to prove the lemma for the average of α under this action, namely

$$\bar{\alpha} = \int_G g^*(\alpha)\,dg \quad \text{where } dg \text{ is Haar measure on } G.$$

The advantage of $\bar{\alpha}$ will be that it must be G-invariant. Note that $\bar{\alpha}$ can be C^∞-approximated by finite averages over suitably chosen finite subsets $\{g_i\}$ of G:

$$\bar{\alpha} \approx \frac{1}{N}\sum_i^N g_i^*(\alpha) \tag{6.6}$$

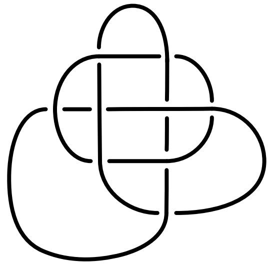


Figure 6.7. The link $9^2_{50} = L9n14$.

and from this it follows that $\overline{\alpha}$ is closed, coclosed, and $\int_C \overline{\alpha} = 1$. Moreover, since all $\left\| g_i^*(\alpha) \right\|_{L^2} = \|\alpha\|_{L^2}$, the triangle inequality applied to (6.6) gives that $\left\|\overline{\alpha}\right\|_{L^2} \leq \|\alpha\|_{L^2}$. This shows we need only consider the G-invariant form $\overline{\alpha}$.

We next show that $\overline{\alpha}$ is equal to $\omega = \frac{1}{\epsilon} dz$; combined with the calculation of $\|\omega\|_{L^2}$ in (6.5), this will establish the lemma. Since the 1-forms $\{dr, d\theta, dz\}$ are G-invariant, it follows that $\overline{\alpha}$ can be expressed as

$$\overline{\alpha} = a(r)\,dr + b(r)\,d\theta + c(r)\,dz$$

Using that $d\overline{\alpha} = 0$, we get that b and c must be constants; as $|d\theta| = 1/\sinh(r)$, we must further have $b = 0$ so that $\overline{\alpha}$ makes sense along the core C where $r = 0$. As $\overline{\alpha}$ is coclosed, we learn that

$$\begin{aligned} 0 = d(*\overline{\alpha}) &= d\big(a(r)\sinh(r)\cosh(r)\,d\theta\wedge dz + c\tanh(r)\,dr\wedge d\theta\big) \\ &= \frac{\partial}{\partial r}\big(a(r)\sinh(r)\cosh(r)\big)\,dr\wedge d\theta\wedge dz \end{aligned}$$

Consequently, $a(r)\sinh(r)\cosh(r)$ is constant. Moreover, that constant must be 0 to prevent $a(r)$ from blowing up as $r \to 0$. So we know $\overline{\alpha} = c\,dz$ and finally that c must be $1/\epsilon$ to ensure $\int_C \overline{\alpha} = 1$. So $\overline{\alpha}$ is equal to ω as claimed, proving the lemma. □

1.6 Theorem. *There exists a sequence of M_n and $\phi_n \in H^1(M_n;\mathbb{R})$ so that*

(a) *The volumes of the M_n are uniformly bounded and* $\operatorname{inj}(M_n) \to 0$ *as $n \to \infty$.*

(b) $\left\|\phi_n\right\|_{L^2} / \left\|\phi_n\right\|_{Th} \to \infty$ *like* $\sqrt{-\log\big(\operatorname{inj}(M_n)\big)}$ *as $n \to \infty$.*

Proof of Theorem 1.6. Our examples are made by Dehn filling a certain 2-cusped hyperbolic 3-manifold. Let W be a compact manifold with $\partial W = T_1 \sqcup T_2$ where both T_i are tori, whose interior is hyperbolic, which fibers over the circle, and where maps $H_1(T_i;\mathbb{Z}) \to H_1(T_i;\mathbb{Z})$ are isomorphisms. (The last condition should be interpreted as saying that W is homologically indistinguishable from $T \times I$.) Further, we require that W has an orientation-preserving involution which interchanges the T_i and acts on $H_1(W;\mathbb{Z})$ by -1. For example, one such W is the exterior of the 2-component link $9^2_{50} = L9n14$ in S^3 shown in Figure 6.7. Since W fibers, there is a 1-dimensional face F of the Thurston norm ball so that any $\phi \in H^1(W;\mathbb{Z})$ in the cone $C_F = \mathbb{R}_{>0} \cdot F$ can be represented by a fibration. Choose $\alpha, \beta \in C_F$ that form an integral basis for $H^1(W;\mathbb{Z})$, and let $a, b \in H_1(W;\mathbb{Z})$ be the dual homological basis where $\alpha(a) = \beta(b) = 1$ and $\alpha(b) = \beta(a) = 0$. Let M_n be the closed manifold obtained by Dehn filling W along the curves in T_i homologically equal to $a - nb$; since W is a $\mathbb{Z}$-homology $T \times I$, it follows that $H^1(M_n;\mathbb{Z})$ is $\mathbb{Z}$ and is generated by the extension ϕ_n of $\widetilde{\phi}_n = n\alpha + \beta$ to M_n. We will show there are constants $c_1, c_2 > 0$ so that

(i) For all n, we have $\|\phi_n\|_{Th} = n\|\alpha\|_{Th} + \|\beta\|_{Th} - 2$.

(ii) For large n, the manifold M_n is hyperbolic. Moreover, we have $\mathrm{vol}(M_n) \to \mathrm{vol}(W)$ and $\mathrm{inj}(M_n) \sim \frac{c_1}{n^2}$ as $n \to \infty$.

(iii) For large n, there is a tube V_n in M_n with core geodesic γ_n of length $2\,\mathrm{inj}(M_n)$, with depth $R_n \geq \operatorname{arcsinh}(c_2 n)$, and where $\int_{\gamma_n} \phi_n = 1$.

Here is why these three claims imply the theorem. From (ii) and (iii), an easy calculation with Lemma 6.2 gives

$$\|\phi_n\|_{L^2} \geq \|\phi_n|_{V_n}\|_{L^2} \geq c_2 n\sqrt{\log n} \quad \text{for all large } n. \tag{6.8}$$

Combining (6.8) with (i-iii) now gives both parts of Theorem 1.6. So it remains to prove the claims (i-iii).

For (i), first note that since $\|\cdot\|_{Th}$ is linear on C_F we have

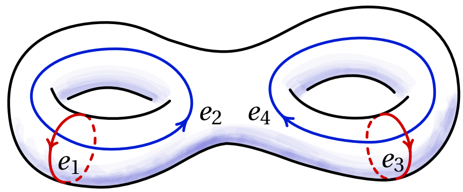
$$\|\widetilde{\phi}_n\|_{Th} = n\|\alpha\|_{Th} + \|\beta\|_{Th}$$

Let $\widetilde{S}_n$ be a fiber in the fibration dual to $\widetilde{\phi}_n$, and hence $\chi_-(\widetilde{S}_n) = \|\widetilde{\phi}_n\|_{Th}$. For homological reasons, the boundary of $\widetilde{S}_n$ has only one connected component in each T_i. Thus $\widetilde{S}_n$ can be capped off with two discs to give a surface $S_n \subset M_n$ which is a fiber in a fibration of M_n over the circle. Hence

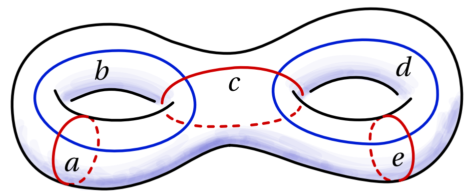
$$\|\phi_n\|_{Th} = \chi_-(S_n) = \chi_-(\widetilde{\phi}_n) - 2 = \|\widetilde{\phi}_n\|_{Th} - 2 = n\|\alpha\|_{Th} + \|\beta\|_{Th} - 2$$

proving claim (i).

Starting in on (ii), the Hyperbolic Dehn Surgery Theorem [Thu1] shows that M_n is hyperbolic for large n and that $\mathrm{vol}(M_n)$ converges to $\mathrm{vol}(W)$. Moreover, for large



(a) Generators of $H_1(S;\mathbb{Z})$



(b) Curves for Dehn twisting

Figure 7.1. Conventions for the genus 2 surface S.

enough n, the cores of the Dehn filling solid tori are isotopic to the two shortest geodesics in M_n. In fact, these core geodesics have the same length: the required involution of W extends to M_n and thus give an isometry of M_n which interchanges them. Let $\gamma_n \subset M_n$ be the core geodesic inside T_1. By [NZ, Proposition 4.3], there is a constant $c_1 > 0$ so that $\mathrm{length}(\gamma_n) \sim \frac{2c_1}{n^2}$ as $n \to \infty$. As $2\,\mathrm{inj}(M_n) = \mathrm{length}(\gamma_n)$, we have shown (ii).

Finally, for (iii), first note that γ_n meets the fiber surface S_n in a single point, giving $\int_{\gamma_n} \phi_n = 1$ if we orient γ_n suitably. So it remains only to estimate the depth R_n of the Margulis tube V_n about γ_n. Let V_n' be the image of V_n under the above involution of M_n. The picture from the Hyperbolic Dehn Surgery Theorem shows that the geometry of M_n outside the $V_n \sqcup V_n'$ converges to that of a compact subset of W. As $\mathrm{vol}(M_n) \to \mathrm{vol}(W)$, we must have that the volume of V_n converges as $n \to \infty$. A straightforward calculation using (6.4) now gives the lower bound on R_n claimed in (iii). □

7 Proof of Theorem 1.8

This section is devoted to proving:

1.8 Theorem. *There exist constants $C_1, C_2, \epsilon_1 > 0$ and a sequence of closed hyperbolic 3-manifolds M_n with $\mathrm{vol}(M_n) \to \infty$ where for all n:*

(a) $\mathrm{inj}(M_n) > \epsilon_1$,

(b) $b_1(M_n) = 1$,

(c) $\|\phi_n\|_{Th} > C_1 e^{C_2 \mathrm{vol}(M_n)}$ *where ϕ_n is a generator of $H^1(M_n;\mathbb{Z})$.*

We first outline the construction of the M_n, sketch why they should have the desired properties, and finally fill in the details. In contrast with the rest of this paper, in this section, all (co)homology groups will use $\mathbb{Z}$ coefficients.

7.2 The construction. Let S be a fixed surface of genus 2. We use the basis e_i for $H_1(S)$ shown in Figure 7.1(a); we take e^i to be the algebraically dual basis of $H^1(S)$, that is, the one where $e^i(e_j) = \delta_{ij}$. Let W be a compact hyperbolic 3-manifold with totally geodesic boundary, where ∂W has genus 2 and the map $H_1(\partial W) \to H_1(W)$ is onto; one possible choice for W is the tripus manifold of [Thu3, §3.3.12]. Note that homologically W is indistinguishable from a genus 2 handlebody. Let W_1 and W_2 be two copies of W whose boundaries have been marked by S so that $\ker(H_1(S) \to H_1(W_1)) = \langle e_2, e_4 \rangle$ and $\ker(H_1(S) \to H_1(W_2)) = \langle e_2, e_3 \rangle$. Thus $H_1(W_1)$ is spanned by the images of e_1 and e_3 and so $H^1(W_1)$ is spanned by natural extensions of e^1 and e^3. For a carefully chosen pseudo-Anosov $f\colon S \to S$, the examples used in Theorem 1.8 are

$$M_n = W_1 \cup_{f^n} W_2 = {W_1 \sqcup W_2}\big/ (x \in \partial W_1) \sim (f^n(x) \in \partial W_2)$$

We first sketch the requirements on f and the overall structure of the argument. Note that our particular markings mean that $H^1(M_0) = \mathbb{Z} = \langle e^1 \rangle$, where we are taking f^0 to be the identity map on S. The key requirement is that $f^*\colon H^1(S) \to H^1(S)$ preserves the subspace $\langle e^1, e^3 \rangle$ and acts on it as an Anosov matrix in $\mathrm{SL}_2\mathbb{Z}$. We will also arrange that $H^1(M_n) = \mathbb{Z} = \langle \phi_n \rangle$ for all n. Since the action of f^* on $\langle e^1, e^3 \rangle$ is complicated, the coefficients of ϕ_n with respect to $\{e^1, e^3\}$ will grow exponentially in n. This will force the restriction of ϕ_n to W_1 to have Thurston norm which is exponential in n, and a standard lemma will then give that ϕ_n itself has Thurston norm which is exponential in n. We will also show that $\mathrm{vol}(M_n)$ grows (essentially) linearly in n, and combining these will give part (c) of Theorem 1.8.

The specific choice we make for f is given in the next lemma.

7.3 Lemma. *There exists a pseudo-Anosov $f\colon S \to S$ whose action f_* on $H_1(S)$ is given by*

$$B = \begin{pmatrix} 3 & 0 & 1 & 0 \\ 1 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 3 \end{pmatrix}$$

Proof. The homomorphism $\mathcal{MCG}(S) \to \mathrm{Aut}(H_1(S))$ induced by taking the action of a mapping class on homology is onto the symplectic group of the form given by

$$J = \left(\begin{array}{cc|cc} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{array}\right)$$

where we are following the conventions given in Figure 7.1(a). It is easy to check that $B^t JB = J$, and so there is some $f \in \mathcal{MCG}(S)$ with $f_* = B$. Composing f with a complicated element of the Torelli group if necessary, we can arrange that f is pseudo-Anosov as required.

Alternatively, consider the following product of Dehn twists:

$$f = \tau_a \circ \tau_d^{-1} \circ \tau_c \circ \tau_b^{-1} \circ \tau_d \circ \tau_c^{-1} \circ \tau_e^{-1}$$

where the curve labeling conventions for the right-handed Dehn twists τ follow Figure 7.1(b). An easy calculation gives $f_* = B$, and using [BHS, CDGW, HIKMOT] one can rigorously verify that the mapping torus of f is hyperbolic with volume ≈ 7.51768989647; in particular, f is pseudo-Anosov. □

The precise technical statements needed to prove Theorem 1.8 are the following three lemmas.

7.4 Lemma. *For all n, the group $H^1(M_n) \cong \mathbb{Z}$ is generated by $\phi_n = a_n e^1 + c_n e^3$ where both a_n and c_n grow exponentially in n at a rate of $\lambda = \frac{3+\sqrt{5}}{2} \approx 2.62$.*

7.5 Lemma. *The manifolds M_n are all hyperbolic with injectivity radius bounded uniformly below, and $\mathrm{vol}(M_n) \asymp n$ as $n \to \infty$.*

7.6 Lemma. *Suppose M^3 is a closed irreducible 3-manifold. Suppose $F \subset M$ is an incompressible surface dividing M into submanifolds A and B. For all $\phi \in H^1(M)$ we have*

$$\|\phi\|_{Th} \geq \|\phi_A\|_{Th} + \|\phi_B\|_{Th}$$

where ϕ_A and ϕ_B are the images of ϕ in $H^1(A)$ and $H^1(B)$ respectively.

We first prove Theorem 1.8 assuming the lemmas, and then establish each of them in turn.

Proof of Theorem 1.8. By Lemma 7.6, if $\overline{\phi}_n$ denotes the restriction of ϕ_n to $H^1(W_1)$, we have

$$\|\phi_n\|_{Th} \geq \left\|\overline{\phi}_n\right\|_{Th}$$

Since W_1 is hyperbolic with totally geodesic boundary, it is atoroidal and acylindrical and hence the Thurston norm on $H^1(W_1)$ is nondegenerate. As any two norms on a finite-dimensional vector space are uniformly comparable, Lemma 7.4 gives that $\left\|\overline{\phi}_n\right\|_{Th} \asymp \lambda^n$ as $n \to \infty$. Since $\mathrm{vol}(M_n) \asymp n$ by Lemma 7.5, we have that $\|\phi_n\|_{Th}$ grows exponentially in $\mathrm{vol}(M_n)$ as required. □

7.7 Remark. In fact, working a little harder one can make the rate of exponential growth explicit, namely

$$\log \|\phi_n\|_{Th} > 0.348 \cdot \mathrm{vol}(M_n) \quad \text{for large } n. \tag{7.8}$$

Specifically, take f to be the map constructed in the second proof of Lemma 7.3. If we use [Tian] in the manner of [Nam, Chapter 12] to get a refined version of the model for M_n, it follows that $\mathrm{vol}(M_n)/n$ limits to $\mathrm{vol}(M_f) \approx 7.51768989$. Combining with the explicit formula for λ in Lemma 7.4 gives (7.8).

Proof of Lemma 7.4. First, we show that $H^1(M_n) = \mathbb{Z}$ for all n, and moreover identify the generator ϕ_n in terms of the basis e^i for $H^1(S)$. (The stronger claim $H_1(M) \cong \mathbb{Z}$ is also true, but we have no need for this here.) Let

$$F = f^* = B^t = \begin{pmatrix} 3 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & -1 & 0 & 3 \end{pmatrix}$$

By Mayer-Vietoris, we have

$$\begin{aligned} H^1(M_n) &= \mathrm{image}\big(H^1(W_1) \to H^1(S)\big) \cap (f^*)^n \Big(\mathrm{image}\big(H^1(W_2) \to H^1(S)\big)\Big) \\ &= \langle e^1, e^3 \rangle \cap \langle F^n(e^1), F^n(e^4) \rangle \end{aligned} \tag{7.9}$$

Notice that F preserves the subspace $\langle e^1, e^3 \rangle$ and acts there by the matrix

$$\overline{F} = \begin{pmatrix} 3 & -1 \\ 1 & 0 \end{pmatrix}$$

which is an Anosov matrix in $\mathrm{SL}_2\mathbb{Z}$. For $n = 0$, the intersection in (7.9) is

$$\langle e^1, e^3 \rangle \cap \langle e^1, e^4 \rangle = \langle e^1 \rangle.$$

Hence, for general n the intersection is spanned by $F^n(e^1)$ which is $a_n e^1 + c_n e^3$ where

$$\overline{F}^n = \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix}$$

Thus $H^1(M_n) = \mathbb{Z}$ as claimed, with generator ϕ_n restricting to $H^1(W_1)$ as $a_n e^1 + c_n e^3$, where a_n and c_n grow exponentially in n, specifically at a rate $\lambda = \frac{3+\sqrt{5}}{2} \approx 2.618034$. □

Proof of Lemma 7.5. Though a geometric limit argument could be given to verify the injectivity radius and volume claims, we refer for efficiency to the main result of [BMNS] for a bi-Lipschitz model for the manifolds M_n. Following the terminology of [BMNS], the *decorated manifolds* $\mathscr{M}$ are the pair of acylindrical manifolds W_1 and W_2 with totally geodesic boundary, the *decorations* μ_1 and μ_2 on the boundaries ∂W_1 and ∂W_2 can be taken to be bounded length markings in the induced hyperbolic structure on each boundary component, and the gluing map is f^n as above. The *large heights* condition in Theorem 8.1 of [BMNS] is evidently satisfied since the curve complex distance

$$d_{\mathscr{C}(\partial W_2)}\big(f^n(\mu_1), \mu_2\big)$$

grows linearly with n and likewise for $\big(\mu_1, f^n(\mu_2)\big)$ in $\mathscr{C}(\partial W_1)$. Furthermore, the pair $\big(f^n(\mu_1), \mu_2\big)$ has R*-bounded combinatorics*, where R is independent of n.

The *model manifold* for the $(\mathscr{M}, R)$-gluing X_n determined by these data is as follows. Let $\widetilde{M}_f$ be the fiber cover of the mapping torus M_f, and let $\widetilde{M}_f[0,1]$ be a fundamental domain for the action of f as an isometric covering translation $\alpha_f : \widetilde{M}_f \to \widetilde{M}_f$ bounded by a choice of fiber and its translate by α_f. Defining $\widetilde{M}_f[k, k+1]$ to be $\alpha_f^k(\widetilde{M}_f[0,1])$, we use $\widetilde{M}_f[0,n]$ to denote the union of n successive such fundamental domains. Then the model manifold $\mathbb{M}_{X_n}$ is the gluing of W_1 and W_2 along their boundary to $\widetilde{M}_f[0,n]$ in the manner described in [BMNS, §2.15]. Given that $\widetilde{M}_f$ is periodic, we know that $\mathrm{inj}(\mathbb{M}_{X_n})$ is bounded below independent of n and that $\mathrm{vol}(\mathbb{M}_{X_n}) \sim \mathrm{vol}(M_f) \cdot n$ as $n \to \infty$.

Now Theorem 8.1 of [BMNS] gives a K so that for all large n there is a K–bi-Lipschitz diffeomorphism

$$f_{X_n} \colon \mathbb{M}_{X_n} \to M_n$$

Combined with the above facts about the geometry of $\mathbb{M}_{X_n}$, this gives the claimed properties for M_n and so proves the lemma. □

Proof of Lemma 7.6. Let $S \subset M$ be a surface dual to ϕ which is *taut*, that is, the surface S realizes $\|\phi\|_{Th}$, is incompressible, and no union of components of S is separating. As F and S are incompressible, we can isotope S so that $F \cap S$ consists of curves that are essential in both S and F; in particular, every component of $S \setminus F$ has non-positive Euler characteristic. As $S \cap A$ and $S \cap B$ are dual to ϕ_A and ϕ_B respectively, we have

$$\|\phi\|_{Th} = -\chi(S) = -\chi(S\cap A) - \chi(S \cap B) = \chi_-(S \cap A) + \chi_-(S\cap B) \geq \|\phi_A\|_{Th} + \|\phi_B\|_{Th}$$

as desired. □

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