

LIMIT SETS OF TEICHMÜLLER GEODESICS WITH MINIMAL NONUNIQUELY ERGODIC VERTICAL FOLIATION, II

JEFFREY BROCK, CHRISTOPHER LEININGER, BABAK MODAMI,
AND KASRA RAFI

ABSTRACT. Given a sequence of curves on a surface, we provide conditions which ensure that (1) the sequence is an infinite quasi-geodesic in the curve complex, (2) the limit in the Gromov boundary is represented by a nonuniquely ergodic ending lamination, and (3) the sequence divides into a finite set of subsequences, each of which projectively converges to one of the ergodic measures on the ending lamination. The conditions are sufficiently robust, allowing us to construct sequences on a closed surface of genus g for which the space of measures has the maximal dimension $3g - 3$, for example.

We also study the limit sets in the Thurston boundary of Teichmüller geodesic rays defined by quadratic differentials whose vertical foliations are obtained from the constructions mentioned above. We prove that such examples exist for which the limit is a cycle the 1-skeleton of the simplex of projective classes of measures visiting every vertex.

CONTENTS

1. Introduction	2
2. Background	4
3. Sequences of curves	11
4. Curve complex quasi-geodesics	12
5. Measures supported on laminations	18
6. Ergodic measures	27
7. Constructions	36
8. Teichmüller geodesics and active intervals	44
9. Limit sets of Teichmüller geodesics	54
References	68

Date: January 12, 2016.

2010 *Mathematics Subject Classification.* Primary 30F60, 32G15, Secondary .

The first author was partially supported by NSF grant DMS-1207572, the second author by NSF grant DMS-1510034, and the fourth author by NCERC grant # 435885.

1. INTRODUCTION

This paper builds on the work of the second and fourth author with Anna Lenzhen, [LLR], in which the authors construct a sequence of curves in the five-punctured sphere S with the following properties (see §2 for definitions). First, the sequence is a quasi-geodesic ray in the curve complex of S , and hence converges to some ending lamination ν . Second, ν is nonuniquely ergodic, and the sequence naturally splits into two subsequences, each of which converges to one of the ergodic measures on ν in the space of projective measured laminations. Third, for any choice of measure $\bar{\nu}$ on ν and base point X in Teichmüller space, the Teichmüller ray based at X and defined by the quadratic differential with vertical foliation $\bar{\nu}$, accumulates on the entire simplex of measures on ν in the Thurston compactification. The construction in [LLR] was actually a family of sequences depending on certain parameters.

In this paper we extract the key features of the sequences produced in the above construction as a set of *local* properties for any sequence of curves $\{\gamma_k\}_{k=0}^\infty$ on any surface, which we denote \mathcal{P} ; see §3 and Definition 3.1 as well as §7 for examples. Here, “local” is more precisely *m-local* for some $2 \leq m \leq \xi(S)$ (where $\xi(S) = \dim_{\mathbb{C}}(\text{Teich}(S))$), and means that the conditions in \mathcal{P} involve relations between curves contained subsets of the form $\{\gamma_k, \dots, \gamma_{k+2m}\}$ for $k \geq 0$. We refer to the number m as the *subsequence counter*. Most of the conditions in \mathcal{P} are stated in terms of intersection numbers, though they also include information about twisting which is recorded in an auxiliary sequence $\{e_k\}_{k=0}^\infty \subset \mathbb{N}$.

Theorem 1.1. *For appropriate choices of parameters in \mathcal{P} , any sequence $\{\gamma_k\}_{k=0}^\infty \subset \mathcal{C}(S)$ satisfying \mathcal{P} will be the vertices of a quasi-geodesic in $\mathcal{C}(S)$ and hence will limit to an ending lamination ν in $\partial\mathcal{C}(S) \cong \mathcal{EL}(S)$.*

If $\mu = \gamma_0 \cup \dots \cup \gamma_{m-1}$, then for any $k \geq m$, the subsequence counter, we have

$$d_{\gamma_k}(\mu, \nu) \stackrel{+}{\asymp} e_k.$$

On the other hand, there is a constant $R > 0$ with the property that for any proper subsurface $W \neq \gamma_k$ for any $k \in \mathbb{N}$ we have

$$d_W(\mu, \nu) < R.$$

See Propositions 4.4 and 4.5 for precise statements. Here d_W is the projection coefficient for W and d_γ the projection coefficient for (the annular neighborhood of) γ ; see §2.4.

Although the conditions in \mathcal{P} only provide local information about intersection numbers, we can deduce estimates on intersection numbers between any two curves in the sequence from this; see Theorem 5.1. From these estimates, we are able to promote the convergence $\gamma_k \rightarrow \nu$ in $\overline{\mathcal{C}(S)}$ into precise information about convergence in $\mathcal{PML}(S)$. To state this, we note that the local condition depends on the subsequence counter m . There are m subsequences $\{\gamma_i^h\}_{i=0}^\infty$, for $h = 0, \dots, m-1$, defined by $\gamma_i^h = \gamma_{im+h}$.

Theorem 1.2. *For appropriate choices of parameters in \mathcal{P} , and any sequence $\{\gamma_k\}_{k=0}^\infty \subset \mathcal{C}(S)$ satisfying \mathcal{P} , the ending lamination $\nu \in \mathcal{EL}(S)$ from Theorem 1.1 is nonuniquely ergodic. Moreover, if m is the subsequence counter, then the dimension of the space of measures on ν is precisely m , and the m subsequences $\{\gamma_i^h\}_{i=0}^\infty$ converge to m ergodic measures $\bar{\nu}^h$ on ν , for $h = 0, \dots, m-1$, spanning the space of measures.*

For precise statements, see Theorems 5.10, 6.1, and 6.7. We note that subsequence counter m can be at most $\xi(S)$ (which is the maximum dimension of the space of measures on any lamination on S). The constructions in §7 are quite flexible and provide examples with this maximal dimension, as well as smaller dimensions.

As an application of these theorems, together with the main result of the first and third author in [BM] and Theorem 1.1, we have.

Corollary 1.3. *Suppose ν is as in Theorem 1.1. Any Weil-Petersson geodesic ray with forward ending lamination ν is recurrent to a compact subset of the moduli space.*

Here, the *ending lamination* of a Weil-Petersson geodesic ray is given as in [BMM10, BMM11]. The Corollary, which follows directly from [BM, Thm. 4.1] after observing that ν satisfies the condition of *nonannular bounded combinatorics* (see Proposition 4.5), provides greater insight into the class of Weil-Petersson ending laminations that violate *Masur's criterion*. In particular, these nonuniquely ergodic laminations determine *recurrent* Weil-Petersson geodesic rays, by contrast to the setting of Teichmüller geodesics where Masur's criterion [Mas92] guarantees a Teichmüller geodesic with such a vertical foliation diverges.

For any lamination ν coming from a sequence $\{\gamma_k\}_{k=0}^\infty$ satisfying \mathcal{P} , as well as some additional conditions (see (8.9) in §8 and condition $\mathcal{P}(iv)$ in §9), we analyze the limit set of a Teichmüller geodesic ray defined by a quadratic differential with vertical foliation $\bar{\nu}$ supported on ν . To describe our result about the limiting behavior of this geodesic ray, we denote the simplex of the projective classes of measures supported on the lamination by $\Delta(\nu)$ in the space of projective measured foliations, viewed as the Thurston boundary of Teichmüller space.

Theorem 1.4. *Suppose that ν is the limiting lamination of a sequence $\{\gamma_k\}_{k=0}^\infty$ satisfying the conditions \mathcal{P} , $\mathcal{P}(iv)$, and (8.9). Let $\bar{\nu} = \sum_{h=0}^{m-1} x_h \bar{\nu}^h$ where $x_h > 0$ for $h = 0, \dots, m-1$, and $r : [0, \infty) \rightarrow \text{Teich}(S)$ be a Teichmüller geodesic ray with vertical measured lamination $\bar{\nu}$. Then the limit set of r in the Thurston boundary is the simple closed curve in the simplex $\Delta(\nu)$ of measures on ν that is the concatenation of edges*

$$[[\bar{\nu}^0], [\bar{\nu}^1]] \cup [[\bar{\nu}^1], \bar{\nu}^2]] \cup \dots \cup [[\bar{\nu}^{m-1}], [\bar{\nu}^0]].$$

When $m \geq 3$, the theorem shows that there are Teichmüller geodesics whose limit set does not contain any point in the interior of $\Delta(\nu)$. In addition, it answers the following question raised by Jonathan Chaika:

Question 1.5. Is the limit set of each Teichmüller geodesic ray simply connected?

For $m \geq 3$, the theorem shows that answer to this question is no. Namely, Teichmüller geodesic rays with vertical measured lamination as above provide examples of geodesics with limit set being a topological circle, and hence not simply connected.

The results of this paper (as well as those of [LLR]) were inspired by work of Masur in [Mas82], Lenzhen [Len08], and Gabai [Gab09]. In [Mas82] Masur showed that if ν is a uniquely ergodic foliation, then any Teichmüller ray defined by a quadratic differential with vertical foliation supported on ν limits to $[\nu]$ in the Thurston compactification. Lenzhen [Len08] gave the first examples of Teichmüller rays which do not converge in the Thurston compactification. Lenzhen's rays were defined by quadratic differentials with non-minimal vertical foliations, and in both [LLR] and [CMW], nonconvergent rays defined by quadratic differentials with minimal vertical foliations were constructed. The methods in these two papers are quite different, and as mentioned above, the approach taken in this paper is more closely related to that of [LLR]. We also note that our example of nonuniquely ergodic laminations obtained from a sequence of curves are similar to those produced by Gabai's in [Gab09]. On the other hand, our construction provides additional information, especially important are the estimates on intersection numbers and subsurface projections, that allow us to study the limiting behavior of the associated Teichmüller rays. For more on the history and results about the existence and constructions of nonuniquely ergodic laminations and the study of limit sets of Teichmüller geodesics with such vertical laminations we refer the reader to the introduction of [LLR].

Acknowledgement: We would like to thank Howard Masur for illuminating conversations and communications. We also would like to thank Anna Lenzhen; her collaboration in the first paper which was crucial for the development of the current paper. Finally we would like to thank MSRI at Berkeley for hosting the program Dynamics on moduli spaces in April 2015; where the authors had the chance to form some of the techniques of this paper.

2. BACKGROUND

We use the following notation throughout this paper.

Notation 2.1. Suppose $K \geq 1$ and $C \geq 0$ and $f, g : X \rightarrow \mathbb{R}$ are two functions. We write $f \overset{+}{\succ}_C g$ if $f(x) - C \leq g(x) \leq f(x) + C$, $f \overset{*}{\succ}_K g$ if $\frac{1}{K}f(x) \leq g(x) \leq Kf(x)$, and $f \succ_{K,C} g$ if $\frac{1}{K}(f(x) - C) \leq g(x) \leq Kf(x) + C$. We also write $f \overset{*}{\prec}_K g$ if $f(x) \leq Kg(x)$, $f \overset{+}{\prec}_C g$ if $f(x) \leq g(x) + C$, and $f \prec_{K,C} g$ if $f(x) \leq Kg(x) + C$. When the constants are known from the

text we drop them from the notations. Finally, we also write $f = O(g)$ if $f \prec^* g$.

Let $S = S_{g,b}$ be an orientable surface of finite type with genus g and b holes (a hole can be either a puncture or a boundary component). Define the complexity of S by $\xi(S) = 3g - 3 + b$. The main surface we will consider will have $\xi > 1$ and all holes will be punctures. However, we will also be interested in subsurfaces and covers of the main surface, which can also have $\xi \leq 1$. For surfaces S with $\xi(S) \geq 1$, we will equip it with a reference metric, which is any complete, hyperbolic metric of finite area with geodesic boundary (if any).

2.1. Curve complexes. For any surface Y , $\xi(Y) \geq 1$, the curve complex of Y , denoted by $\mathcal{C}(Y)$, is a flag complex whose vertices are the isotopy classes of simple closed curves on Y that are essential, meaning non-null homotopic and non-peripheral. For $\xi(Y) > 1$, a set of $k + 1$ distinct isotopy classes of curves defines a k -simplex if any pair can be represented by disjoint curves. For $\xi(Y) = 1$ (Y is $S_{0,4}$ or $S_{1,1}$), the definition is modified as follows: a set of $k + 1$ distinct isotopy classes defines a k -simplex if the curves can be represented intersecting twice (for $Y = S_{0,2}$) or once (for $Y = S_{1,1}$).

The only surface Y with $\xi(Y) < 1$ of interest for us is a compact annulus with two boundary components. These arise as follows. For any essential simple closed curve α on our main surface S , let Y_α denote the annular cover of S to which α lifts. The reference hyperbolic metric on S lifts and provides a compactification of this cover by a compact annulus with boundary (which is independent of the metric). The curve complex of α , denoted $\mathcal{C}(Y_\alpha)$, or simply $\mathcal{C}(\alpha)$, has vertex set being the properly embedded, essential arcs in Y_α , up to isotopy fixing the boundary pointwise. A set of isotopy classes of arcs spans a simplex if any pair can be realized with disjoint interiors.

Distances between vertices in $\mathcal{C}(Y)$ (for any Y) will be measured in the 1-skeleton, so the higher dimensional simplices are mostly irrelevant. Masur and Minsky [MM99] proved that for any Y , there is a $\delta > 0$ so that $\mathcal{C}(Y)$ is δ -hyperbolic.

For surfaces Y with $\xi(Y) \geq 1$, we also consider the arc and curve complex $\mathcal{AC}(Y)$, defined in a similar way to $\mathcal{C}(Y)$. Here vertices are isotopy classes of essential simple closed curves and essential, properly embedded arcs (isotopies need not fix the boundary pointwise), with simplices defined again in terms of disjoint representatives. Arc and curve complexes are quasi-isometric to curve complexes, and so are also δ -hyperbolic.

Multicurves (respectively, multiarcs) are disjoint unions of pairwise non-isotopic essential simple closed curves (respectively, simple closed curves and properly embedded arcs). Up to isotopy a multicurve (respectively, multiarc) determines, and is determined by, a simplex in $\mathcal{C}(S)$ (respectively, $\mathcal{AC}(S)$). A marking μ is a pants decomposition $\text{base}(\mu)$, called the base of μ , together with a transversal curve β_α , for each $\alpha \in \text{base}(\mu)$, which is a curve minimally intersecting α and disjoint from $\text{base}(\mu) - \alpha$. A partial marking μ

is similarly defined, but not every curve in the pants decomposition base(μ) is required to have a transversal curve.

For more details on curve complexes, arc and curve complexes, and markings, we refer the reader to [MM99].

Remark 2.2. When the number $\xi(S)$ is at least 1, it is equal to the number of curves in a pants decomposition. When all the holes of S are punctures, $\xi(S)$ is also the complex dimension of Teichmüller space of S .

2.2. Laminations and foliations. A lamination will mean a geodesic lamination (with respect to the reference metric if no other metric is specified), and a measured lamination is a geodesic lamination ν , called the support, with an invariant transverse measure $\bar{\nu}$. We will often refer to a measured lamination just by the measure $\bar{\nu}$ (as this determines the support ν). The space of all measured laminations will be denoted $\mathcal{ML}(S)$, and for any two metrics, the resulting spaces of measured laminations are canonically identified. By taking geodesic representatives, simple closed curves and multicurves determine geodesic laminations. Weighted simple closed curves and multicurves determine measured laminations are dense in $\mathcal{ML}(S)$, and the geometric intersection number extends to a continuous, bi-homogeneous function

$$i: \mathcal{ML}(S) \times \mathcal{ML}(S) \rightarrow \mathbb{R}.$$

By a measured foliation on S we will mean a singular measured foliation with prong singularities of negative index (and at punctures, filling in the puncture produces a k -prong singularity with $k \geq 1$). When convenient, a measured foliation may be considered only defined up to measure equivalence, and the space of measure equivalence classes of measured foliations is denoted $\mathcal{MF}(S)$. The spaces $\mathcal{MF}(S)$ and $\mathcal{ML}(S)$ are canonically identified, and we will frequently not distinguish between measured laminations and measured foliations. A foliation or lamination is uniquely ergodic if it supports a unique (up to scaling) transverse measure, or equivalently, if the first return map to (the double of) any transversal is uniquely ergodic. Otherwise it is nonuniquely ergodic. We write $\mathcal{PML}(S)$ and $\mathcal{PMF}(S)$ for the quotient spaces, identifying measured laminations or foliations that differ by scaling the measure. See [PH92, CEG87, FLP79, Thu86, Lev83] for complete definitions, detailed discussion, and equivalence of $\mathcal{MF}(S)$ and $\mathcal{ML}(S)$.

2.3. Gromov boundary of the curve complex. A lamination ν on S is called an ending lamination if it is minimal (every leaf is dense) and filling (every simple closed geodesic on the surface nontrivially, transversely intersect ν). Every ending lamination admits a transverse measure, and we let $\mathcal{EL}(S)$ denote the space of all ending laminations. This is the quotient space of the subspace of $\mathcal{ML}(S)$ consisting of measured laminations supported on

ending laminations, by the map which forgets the measures. The following theorem of Klarreich [Kla] identifies the Gromov boundary of $\mathcal{C}(S)$ with $\mathcal{EL}(S)$.

Theorem 2.3. (Boundary of the curve complex) *There is a homeomorphism Φ from the Gromov boundary of $\mathcal{C}(S)$ equipped with its standard topology to $\mathcal{EL}(S)$.*

Let $\{\gamma_k\}_{k=0}^\infty$ be a sequence of curves in $\mathcal{C}_0(S)$ that converges to a point x in the Gromov boundary of $\mathcal{C}(S)$. Regarding each γ_k as a projective measured lamination, any accumulation point of the sequence $\{\gamma_k\}_{k=0}^\infty$ in $\mathcal{PML}(S)$ is supported on $\Phi(x)$.

We will use this theorem throughout to identify points in $\partial\mathcal{C}(S)$ with ending laminations in $\mathcal{EL}(S)$.

2.4. Subsurface coefficients. An *essential subsurface* Y of a surface Z with $\xi(Y) \geq 1$ is a closed, connected, embedded subsurface whose boundary components are either essential curves in Z or boundary component of Z , and whose punctures are punctures of Z . All such subsurfaces are considered up to isotopy, and we often choose representatives that are components of complements of small neighborhoods of simple closed geodesics, which therefore have minimal, transverse intersection with any lamination. The only essential subsurfaces Y of Z with $\xi(Y) < 1$ are not actually subsurfaces at all, but rather such a Y is the compactified annular covers Y_α of Z associated to a simple closed curve α in Z . We sometimes confuse an annular neighborhood of α with the cover Y_α (hence the reference to it as a subsurface) when convenient. We will always write $Y \subseteq Z$ to denote an essential subsurface, even when it is not, strictly speaking, a subset of Z .

Let $Y \subseteq Z$ be an essential non-annular subsurface and λ a lamination (possibly a multicurve) and we define the subsurface projection of λ to Y . Represent Y as a component of the complement of a very small neighborhood of geodesic multicurve. If $\lambda \cap Y = \emptyset$, then define $\pi_Y(\lambda) = \emptyset$. Otherwise, $\pi_Y(\lambda)$ is the union of all curves which are (i) simple closed curve components of $Y \cap \lambda$ or (ii) essential components of $\partial N(a \cup \partial Y)$, where $a \subset \lambda \cap Y$ is any arc, and $N(a \cup \partial Y)$ is a regular neighborhood of the union. If Y_α is an essential annular subsurface, then $\pi_{Y_\alpha}(\lambda)$, or simply $\pi_\alpha(\lambda)$, is defined as follows. For any component of the preimage of λ in the annular cover corresponding to α , the closure is an arc in Y_α , and we take the union of all such arcs that are essential (that is, the arcs that connect the two boundary components).

For a marking μ (or partial marking), if $Y = Y_\alpha$ is an annulus with core curve $\alpha \in \text{base}(\mu)$, then $\pi_Y(\mu) = \pi_\alpha(\beta_\alpha)$, where β_α is the transverse curve for α in μ . Otherwise, $\pi_Y(\mu) = \pi_Y(\text{base}(\mu))$. For any lamination or partial marking λ and any essential subsurface Y , $\pi_Y(\lambda)$ is a subset of diameter at most 2.

Let μ, μ' be laminations, multiarcs, or partial markings on Z and $Y \subset Z$ an essential subsurface. The *Y -subsurface coefficient* of μ and μ' is defined

by

$$(2.1) \quad d_Y(\mu, \mu') := \text{diam}_{\mathcal{C}(Y)}(\pi_Y(\mu) \cup \pi_Y(\mu'))$$

Remark 2.4. The subsurface coefficient is sometimes alternatively defined as the (minimal) distance between $\pi_Y(\mu)$ and $\pi_Y(\mu')$. Since the diameter of the projection of any marking or lamination is bounded by 2, these definitions differ by at most 4. The definition we have chosen satisfies a triangle inequality (when the projections involved are nonempty), which is particularly useful for our purposes.

The following lemma provides an upper bound for a subsurface coefficient in terms of intersection numbers.

Lemma 2.5. [MM00, §2] *Given curves $\alpha, \beta \in \mathcal{C}(S)$, for any essential subsurface $Y \subseteq S$ we have*

$$d_Y(\alpha, \beta) \leq 2i(\alpha, \beta) + 1.$$

When Y is an annular subsurface the above bound holds with multiplicative factor 1.

Remark 2.6. The bound in the above lemma can be improved to $\prec \log i(\alpha, \beta)$ for $\xi(Y) \geq 1$, but the bound given is sufficient for our purposes.

The following result equivalent to [CR07, Corollary D] provides for a comparison between the logarithm of intersection number and sum of subsurfaces coefficients. For a pair of markings μ, μ' , the intersection number $i(\mu, \mu')$ is defined to be the sum of the intersection numbers of the curves in μ with those in μ' .

Theorem 2.7. *Given $A > 0$ sufficiently large, there are constants so that for any two multi-curves, multi-arcs or markings μ and μ' we have*

$$\log i(\mu, \mu') \asymp \sum_{\substack{W \subseteq Y, \\ \text{non-annular}}} \{d_W(\mu, \mu')\}_A + \sum_{\substack{W \subseteq Y, \\ \text{annular}}} \log \{d_W(\mu, \mu')\}_A.$$

Where W is so that $\mu, \mu' \pitchfork W$.

In this theorem, $\{\cdot\}_A$ is a cut-off function defined by $\{x\}_A = x$ if $x \geq A$, and $\{x\}_A = 0$ if $x < A$.

Notation 2.8. Given a lamination or a partial marking μ and subsurface Y we say that μ and Y overlap, writing $\mu \pitchfork Y$ if $\pi_Y(\mu) \neq \emptyset$. For any marking μ and any subsurface Y , we have $\mu \pitchfork Y$. Given two subsurfaces Y and Z , if $\partial Y \pitchfork Z$ and $\partial Z \pitchfork Y$ then we say that Y and Z overlap, and write $Y \pitchfork Z$.

The inequality first proved by J. Behrstock [Beh06] relates subsurface coefficients for overlapping subsurfaces.

Theorem 2.9 (Behrstock inequality). *There is a constant $B_0 > 0$ so that given a partial marking or lamination μ and subsurfaces Y and Z satisfying $Y \pitchfork Z$ we have*

$$\min\{d_Y(\partial Z, \mu), d_Z(\partial Y, \mu)\} \leq B_0.$$

whenever $\mu \pitchfork Y$ and $\mu \pitchfork Z$.

Remark 2.10. As shown in [Man13], the constant B_0 can be taken to be 10. In fact, if one projection is at least 10, then the other is ≤ 4 .

The following theorem is a straightforward consequence of the Bounded Geodesic Image Theorem [MM00, Theorem 3.1].

Theorem 2.11. (Bounded geodesic image) *Given $k \geq 1$ and $c \geq 0$, there is a constant $G > 0$ with the following property. Let $Y \subsetneq S$ be a subsurface. Let $\{\gamma_k\}_{i=0}^\infty$ be a 1-Lipschitz (k, c) -quasi-geodesic in $\mathcal{C}(S)$ so that $\gamma_k \pitchfork Y$ for all i . Then $\text{diam}_Y(\{\gamma_k\}_{i=0}^\infty) \leq G$.*

2.5. Teichmüller theory. We assume that any holes of S are punctures. The Teichmüller space of S , $\text{Teich}(S)$, is the space of equivalence classes of marked complex structures $[f: S \rightarrow X]$, where f is an orientation preserving homeomorphism to a finite type Riemann surface X , where $(f: S \rightarrow X) \sim (g: S \rightarrow Y)$ if $f \circ g^{-1}$ is isotopic to a conformal map. We often abuse notation, and simply refer to X as a point in Teichmüller space, with the equivalence class of marking implicit. We equip $\text{Teich}(S)$ with the Teichmüller metric, whose geodesics are defined in terms of quadratic differentials.

Let X be a finite type Riemann surface and let $T^{(1,0)*}X$ be the holomorphic cotangent bundle of X . A quadratic differential q is a nonzero, integrable, holomorphic section of the bundle $T^{(1,0)*}X \otimes T^{(1,0)*}X$. In local coordinates q has the form $q(z)dz^2$ where $q(z)$ is holomorphic function. Changing to a different coordinate w , q changes by the square of the derivative, and is thus given by $q(z(w))(\frac{\partial w}{\partial z})^2 dw^2$. The integrability condition is only relevant when X has punctures, in which case it guarantees that q has at worst simple poles at the punctures.

In local coordinates away from zeros of q the quadratic differential q determines the 1-form $\sqrt{q(z)}dz^2$. Integrating this 1-form determines a natural coordinate $\zeta = \xi + i\eta$. Then the trajectories of $d\xi \equiv 0$ and $d\eta \equiv 0$, respectively, determine the horizontal and vertical foliations of q on X . Integrating $|d\xi|$ and $|d\eta|$ determines transverse measures on vertical and horizontal foliations, respectively. These extend to measured foliations on the entire surface S with singularities at the zeros. Using the identification $\mathcal{MF}(S) \cong \mathcal{ML}(S)$, we often refer to the vertical and horizontal measured laminations of q .

Now given a quadratic differential q on X , the associated Teichmüller geodesic is determined by the family of Riemann surfaces X_t defined by local coordinates $\zeta_t = e^t\xi + e^{-t}\eta$ where $\zeta = \xi + i\eta$ is a natural coordinate of q at X and $t \in \mathbb{R}$. Every Teichmüller geodesic ray based at X is determined by a quadratic differential q on X . See [Gar87] for details on Teichmüller space and quadratic differentials.

2.6. The Thurston compactification. Given a point $[f: S \rightarrow X]$ in $\text{Teich}(S)$ and a curve α , the hyperbolic length of α at $[f: S \rightarrow X]$ is defined to be hyperbolic length of the geodesic homotopic to $f(\alpha)$ in X . Again

abusing notation and denoting the point in $\text{Teich}(S)$ by X , we write the hyperbolic length simply as $\text{Hyp}_X(\alpha)$. The hyperbolic length function extends to a continuous function

$$\text{Hyp}_{(\cdot)}(\cdot): \text{Teich}(S) \times \mathcal{ML}(S) \rightarrow \mathbb{R}.$$

The Thurston compactification, $\overline{\text{Teich}(S)} = \text{Teich}(S) \cup \mathcal{PM}\mathcal{L}(S)$ is constructed so that a sequence $\{X_n\} \subseteq \text{Teich}(S)$ converges to $[\bar{\nu}] \in \mathcal{PM}\mathcal{L}(S)$ if and only if

$$\lim_{n \rightarrow \infty} \frac{\text{Hyp}_{X_n}(\alpha)}{\text{Hyp}_{X_n}(\beta)} = \frac{i(\bar{\nu}, \alpha)}{i(\bar{\nu}, \beta)}$$

for all simple closed curves α, β with $i(\bar{\nu}, \beta) \neq 0$. See [Bon88, FLP79] for more details on the Thurston compactification.

2.7. Some hyperbolic geometry. Here we list a few important hyperbolic geometry estimates. For a hyperbolic metric $X \in \text{Teich}(S)$ and a simple closed curve α , in addition to the length $\text{Hyp}_X(\alpha)$, we also have the quantity $w_X(\alpha)$, the *width of α in X* . This is the width of a maximal embedded tubular neighborhood of α in the hyperbolic metric X —that is, $w_X(\alpha)$ is the maximal w so that the open $w/2$ -neighborhood of α is an annular neighborhood of α . The Collar Lemma (see e.g. [Bus10, §4.1]) provides a lower bound on the width

Lemma 2.12. *For any simple closed curve α , we have*

$$w_X(\alpha) \geq 2 \sinh^{-1} \left(\frac{1}{\sinh(\text{Hyp}_X(\alpha)/2)} \right).$$

Consequently,

$$(2.2) \quad w_X(\alpha) \stackrel{\pm}{\asymp} 2 \log \left(\frac{1}{\text{Hyp}_X(\alpha)} \right)$$

The second statement comes from the first, together with an easy area argument. The implicit additive error depends only on the topology of S .

We also let $\epsilon_0 > 0$ be the Margulis constant, which has the property that any two hyperbolic geodesics of length at most ϵ_0 must be embedded and disjoint.

2.8. Short markings. For $L > 0$ sufficiently large, an L -bounded length marking at $X \in \text{Teich}(S)$ (or L -short marking) is a marking with the property that any curve in $\text{base}(\mu)$ has hyperbolic length less than L , and so that for each $\alpha \in \text{base}(\mu)$, the transversal curve to α has smallest possible length in X . Choosing ϵ sufficiently large (larger than the Bers constant of the surface) the distance between any two points in Teichmüller space can be estimated up to additive and multiplicative error in terms of the subsurface coefficients of the short markings at those points, together with the lengths of their base curves; see [Raf07].

3. SEQUENCES OF CURVES

Over the course of the next three sections we will provide general conditions on a sequence of curves which guarantee that any accumulation point in $\mathcal{PML}(S)$ of this sequence is a nonuniquely ergodic ending lamination. In [Gab09, §9], Gabai describes a construction of minimal filling nonuniquely ergodic geodesic laminations. The construction is topological in nature. Our construction in this paper and that of [LLR] can be considered as quantifications of Gabai's construction where the estimates for intersection numbers are computed explicitly. These estimates allow us to provide more detailed information about the limits in $\mathcal{PML}(S)$ as well as limiting behavior of associated Teichmüller geodesics.

In this section we state our conditions on the sequence of curves, and describe how to think about them.

Throughout the rest of this paper $\{e_k\}_{k=0}^\infty$ is an increasing sequence of integers satisfying

$$(3.1) \quad e_{k+1} \geq ae_k \text{ for any } k \geq 0.$$

where $a > 1$. Consequently, for all $l < k$, we have $e_k \geq a^{k-l}e_l$. We also fix positive integers $b_1 \leq b \leq b_2$. We will also assume that $e_0 > E + G$ (and hence $e_k > a^k(E + G)$ for all k), where G is the constant from Theorem 2.11 and E is the constant in Theorem 4.1 below.

In the next definition, \mathcal{D}_γ is the Dehn twist in the curve γ .

Definition 3.1. Fix positive integers $b_1 \leq b \leq b_2$ and $m \leq \xi(S)$, as well as a real number $a > 1$ and sequence $\{e_k\}_{k=0}^\infty$ satisfying (3.1) with respect to a . We say that a sequence of curves $\{\gamma_k\}_{k=0}^\infty$ on S satisfies \mathcal{P} if the following properties hold for all $k \geq 0$:

- (i) $\gamma_k, \dots, \gamma_{k+m-1}$ are pair-wise disjoint and distinct,
- (ii) $\gamma_k, \dots, \gamma_{k+2m-1}$ fill the surface S , and
- (iii) $\gamma_{k+m} = \mathcal{D}_{\gamma_k}^{e_k}(\gamma'_{k+m})$, where γ'_{k+m} is a curve such that

$$i(\gamma'_{k+m}, \gamma_j) \begin{cases} \in [b_1, b_2] & \text{for } j \in \{k-m, \dots, k-1\} \\ = b & \text{for } j = k \\ = 0 & \text{for } j \in \{k+1, \dots, k+m-1\} \end{cases}$$

(here we ignore any situation with $j < 0$).

We will wish to impose some additional constraints on the constant a (specifically, we will require it to be chosen so that (5.6) holds), and so in the notation we sometimes express the dependence on a writing $\mathcal{P} = \mathcal{P}(a)$. Of course, \mathcal{P} depends on the choice of constants $b_1 \leq b \leq b_2$ and the sequence $\{e_k\}$, but we will impose no further constraints on the b constants, and the conditions on $\{e_k\}$ depend on a .

Before proceeding, we elaborate a bit on the properties in \mathcal{P} . First we make a simple observation.

Lemma 3.2. *For every $j, k \geq 0$ with $j \in \{k - m + 1, \dots, k\}$, we have $i(\gamma_{k+m}, \gamma_j) \in [b_1, b_2]$ with $i(\gamma_{k+m}, \gamma_k) = b$.*

Proof. Since $\gamma_{k+m} = \mathcal{D}_{\gamma_k}^{e_k}(\gamma'_{k+m})$ and $\mathcal{D}_{\gamma_k}(\gamma_j) = \gamma_j$ (because $i(\gamma_j, \gamma_k) = 0$), it follows that

$$i(\gamma_{k+m}, \gamma_j) = i(\mathcal{D}_{\gamma_k}^{-e_k}(\gamma_{k+m}), \mathcal{D}_{\gamma_k}^{-e_k}(\gamma_j)) = i(\gamma'_{k+m}, \gamma_j) \in [b_1, b_2]$$

proving the first statement. For the special case $j = k$, $i(\gamma'_{k+m}, \gamma_k) = b$, and the second statement follows. \square

Visualizing the properties of \mathcal{P} : The conditions imposed in \mathcal{P} involve intervals of length m and $2m$, as well as mod m congruence conditions. It is useful to view the tail of the sequence starting at any curve γ_i (for example, when $i = 0$ this is the entire sequence), in the following form

$$(3.2) \quad \begin{array}{ccccccc} \gamma_i & \longrightarrow & \gamma_{i+1} & \longrightarrow & \cdots & \longrightarrow & \gamma_{i+m-1} \\ \longleftarrow & & & & & & \longleftarrow \\ \gamma_{i+m} & \longrightarrow & \gamma_{i+m+1} & \longrightarrow & \cdots & \longrightarrow & \gamma_{i+2m-1} \\ \longleftarrow & & & & & & \longleftarrow \\ \gamma_{i+2m} & \longrightarrow & \gamma_{i+2m+1} & \longrightarrow & \cdots & & \end{array}$$

From the first condition of \mathcal{P} , all curves in any row are pairwise disjoint. Lemma 3.2 tells us that γ_i intersects the curve directly below it b times and it intersects everything in the row directly below it between b_1 and b_2 times. The second condition in \mathcal{P} tells us that any two consecutive rows fill S . The third condition (part of which is used in the proof of Lemma 3.2), can be thought of as saying that going straight down two rows from γ_i to γ_{i+2m} gives a curve that “almost” differs by the power of the Dehn twist $\mathcal{D}_{\gamma_{i+m}}^{e_{i+m}}$. To understand this interpretation, note that γ'_{i+2m} and γ_{i+2m} differ precisely by this power of a twist, while on the other hand, each of γ'_{i+2m} and γ_i have intersection number at most b_2 with the filling set $\gamma_i, \dots, \gamma_{i+2m-1}$ (which we view as saying that γ_i and γ'_{i+2m} are “similar”).

4. CURVE COMPLEX QUASI-GEODESICS

The purpose of this section is to provide general conditions (Theorem 4.1) on a sequence of subsurfaces in terms of subsurface coefficients of consecutive elements which guarantee that their boundaries define a quasi-geodesic in the curve complex of the surface. Appealing to Theorem 2.3, we deduce that such sequences determine an ending lamination. We end by proving that a sequence of curves satisfying \mathcal{P} are core curves of annuli satisfying the conditions of Theorem 4.1, and hence are vertices of a quasi-geodesic in $\mathcal{C}(S)$ defining an ending lamination $\nu \in \mathcal{EL}(S)$.

Variations of this result appeared in [Man13], [CLM12], [Mod15], [LLR], and [BM] for example. Here our conditions only involve the intersection

pattern and projection coefficients of fixed number of consecutive subsurfaces along the sequence. In this sense these are local conditions.

Theorem 4.1. (Local to Global) *For any surface S , there are constants $E > C > 0$ with the following properties. Let $\{Y_k\}_{k=0}^\infty$ be a sequence of subsurfaces of S . Suppose that for each integer $k \geq 0$,*

- (1) *The multi-curves $\partial Y_k, \dots, \partial Y_{k+m-1}$ are pairwise disjoint,*
- (2) *$Y_k \cap Y_j$ for all $j \in \{k+m, \dots, k+2m-1\}$, and*
- (3) *$d_{Y_k}(\partial Y_j, \partial Y_{j'}) > E$ for any $j \in \{k+m, \dots, k+2m-1\}$ and any $j' \in \{k-2m+1, \dots, k-m\}$.*

Then for each $j \geq k+m$ and $j' \leq k-m$ we have

$$(4.1) \quad Y_k \cap Y_j \text{ and } Y_k \cap Y_{j'}$$

and

$$(4.2) \quad d_{Y_k}(\partial Y_j, \partial Y_{j'}) \geq d_{Y_k}(\partial Y_{k-m}, \partial Y_{k+m}) - C.$$

Furthermore suppose that for some $n \geq 1$

- (4) *the multi-curves $\partial Y_k, \dots, \partial Y_{k+2n-1}$ fill S .*

Then for any two indices $k, j \geq 0$ with $|k-j| \geq 2n$ we have

$$(4.3) \quad d_S(\partial Y_j, \partial Y_k) \geq \frac{|k-j|}{4n} - \left(\frac{m}{2n} + 1\right).$$

In the hypotheses (as well as the conclusions) of this theorem, we ignore any condition in which there is a negative index.

Proof. Set the constants

$$C = 2B_0 + 4 + G \quad \text{and} \quad E = C + B_0 + G + 4.$$

Here B_0 is the constant from Theorem 2.9 (Behrstock inequality) and G is the constant from Theorem 2.11 (Bounded geodesic image theorem) for a geodesic (i.e. $k = 1, c = 0$). We prove (4.1) and (4.2) simultaneously by a double induction on $(j-k, k-j')$.

For the base of induction, suppose that $m \leq k-j' \leq 2m-1$ and $m \leq j-k \leq 2m-1$. The statement (4.1) follows from (2). To prove (4.2) note that by (1) $\partial Y_{k+m}, \dots, \partial Y_j$ are pairwise disjoint and have non-empty projections to Y_k . Consequently, the distance in Y_k between any two of these boundaries is at most 2, and so $\text{diam}_{Y_k}(\{\partial Y_l\}_{l=k+m}^j) \leq 2$. Similarly, $\text{diam}_{Y_k}(\{\partial Y_l\}_{l=j'}^{k-m}) \leq 2$. By the triangle inequality we have

$$\begin{aligned} d_{Y_k}(\partial Y_j, \partial Y_{j'}) &\geq d_{Y_k}(\partial Y_{k-m}, \partial Y_{k+m}) - d_{Y_k}(\partial Y_j, \partial Y_{k+m}) - d_{Y_k}(\partial Y_{k-m}, \partial Y_{j'}) \\ &\geq d_{Y_k}(\partial Y_{k-m}, \partial Y_{k+m}) - 4 \geq d_{Y_k}(\partial Y_{k-m}, \partial Y_{k+m}) - C \end{aligned}$$

which is the bound (4.2).

Suppose that (4.1) and (4.2) hold for all $m \leq k-j' \leq 2m-1$ and $m \leq j-k \leq n$, where $n \geq 2m-1$. We suppose $j-k = n+1$ and we must prove both (4.1) and (4.2) for $(j-k, k-j')$.

From the base of induction we already have $Y_k \pitchfork Y_{j'}$. To complete the proof of (4.1), we prove $Y_k \pitchfork Y_j$. Since $m = (k+m) - k \leq 2m - 1$ and $m \leq j - (k+m) = n + 1 - m \leq n$, from the inductive hypothesis we have

$$Y_k \pitchfork Y_{k+m} \text{ and } Y_j \pitchfork Y_{k+m}$$

and

$$d_{Y_{k+m}}(\partial Y_k, \partial Y_j) \geq d_{Y_{k+m}}(\partial Y_k, \partial Y_{k+2m}) - C \geq E - C \geq 4.$$

Consequently, $i(\partial Y_k, \partial Y_j) \neq 0$ and $Y_k \pitchfork Y_j$ as required.

We now turn to the proof of (4.2). Since $Y_k \pitchfork Y_j$ and $Y_k \pitchfork Y_{j'}$, by (2) we may write the following triangle inequality

$$(4.4) \quad d_{Y_k}(\partial Y_{j'}, \partial Y_j) \geq d_{Y_k}(\partial Y_{k-m}, \partial Y_{k+m}) - d_{Y_k}(\partial Y_{k-m}, \partial Y_{j'}) - d_{Y_k}(\partial Y_j, \partial Y_{k+m}).$$

Since $m \leq j - (k+m) = n + 1 - m \leq n$, from the inductive hypothesis we have

$$d_{Y_{k+m}}(\partial Y_k, \partial Y_j) \geq d_{Y_{k+m}}(\partial Y_k, \partial Y_{k+2m}) - C \geq E - C \geq B_0.$$

By Theorem 2.9, $d_{Y_k}(\partial Y_{k+m}, Y_j) \leq B_0$. On the other hand, as in the proof of the base case of induction, since $m \leq k - j' \leq 2m - 1$ we have

$$d_{Y_k}(\partial Y_{k-m}, \partial Y_{j'}) \leq 2.$$

Combining these two inequalities with (4.4), we obtain

$$\begin{aligned} d_{Y_k}(\partial Y_{j'}, \partial Y_j) &\geq d_{Y_k}(\partial Y_{k-m}, \partial Y_{k+m}) - B_0 - 2 \\ &\geq d_{Y_k}(\partial Y_{k-m}, \partial Y_{k+m}) - C. \end{aligned}$$

This completes the first half of the double induction.

We now know that (4.1) and (4.2) hold for all j, j', k with $m \leq k - j' \leq 2m - 1$ and all $j - k \geq m$. We assume that they hold for $m \leq k - j' \leq n$ and $j - k \geq m$ where $n \geq 2m - 1$, and prove that they hold for $k - j' = n + 1$. The proof of (4.1) is completely analogous to the proof in the first part of the induction, and we omit it. The proof of (4.2) is also similar, but requires one additional step so we give the proof.

We may again write the triangle inequality (4.4). Since $m \leq (k-m) - j' = n + 1 - m \leq n$ by the inductive hypothesis we have

$$d_{Y_{k-m}}(\partial Y_k, \partial Y_{j'}) \geq E - C \geq B_0,$$

and so Theorem 2.9 again implies $d_{Y_k}(\partial Y_{k-m}, \partial Y_{j'}) \leq B_0$. If $j - k \leq 2m - 1$, then as above $d_{Y_k}(\partial Y_{k+m}, \partial Y_j) \leq 2$. Otherwise, by induction we have

$$d_{Y_{k+m}}(\partial Y_k, \partial Y_j) \geq E - C \geq B_0$$

and Theorem 2.9 once again implies $d_{Y_k}(\partial Y_{k+m}, \partial Y_j) \leq B_0$. Combining these inequalities with (4.4) we have

$$\begin{aligned} d_{Y_k}(\partial Y_{j'}, \partial Y_j) &\geq d_{Y_k}(\partial Y_{k-m}, \partial Y_{k+m}) - B_0 - \max\{2, B_0\} \\ &\geq d_{Y_k}(\partial Y_{k-m}, \partial Y_{k+m}) - C. \end{aligned}$$

This completes the proof of (4.2), and hence the double induction is finished.

Now further assuming (4) we prove (4.3). Note that we must have $n \geq m$. Without loss of generality we assume $j < k$, so that $k - j \geq 2n \geq 2m$. For the rest of the proof, for any $s, r \in \mathbb{Z}$, $s \leq r$, we write $[s, r] = \{t \in \mathbb{Z} \mid s \leq t \leq r\}$.

Suppose δ is any multi-curve. Let $\mathcal{I}(\delta) = \{s \in [j, k] \mid i(\delta, \partial Y_s) \neq 0\}$.

Claim 4.2. Suppose $s', r' \in [j, k] \setminus \mathcal{I}(\delta)$. Then $|r' - s'| \leq 4n - 2$.

Observe that by the claim, $[j, k] \setminus \mathcal{I}(\delta)$ contains fewer than $4n$ integers.

Proof. Without loss of generality, we assume $s' < r'$, and suppose for a contradiction $r' - s' \geq 4n - 1$. By (4) $\partial Y_{s'+n}, \dots, \partial Y_{s'+3n-1}$ fills S , and so there exists $s' + n \leq t \leq s' + 3n - 1$ with $t \in \mathcal{I}(\delta)$.

Now observe that $s' + m \leq s' + n \leq t$ and $t \leq s' + 3n - 1 \leq r' - n \leq r' - m$, by the first part of the theorem we know that

$$d_{Y_t}(\partial Y_{s'}, \partial Y_{r'}) \geq E - C > 4.$$

On the other hand, since $i(\delta, \partial Y_{s'}) = 0 = i(\delta, \partial Y_{r'})$, and since $t \in \mathcal{I}(\delta)$ implies $\pi_{Y_t}(\delta) \neq \emptyset$, the triangle inequality implies

$$d_{Y_t}(\partial Y_{s'}, \partial Y_{r'}) \leq d_{Y_t}(\partial Y_{s'}, \delta) + d_{Y_t}(\delta, \partial Y_{r'}) \leq 2 + 2 = 4$$

a contradiction. \square

Let η be a geodesic in $\mathcal{C}(S)$ connecting ∂Y_j to ∂Y_k . For any $l \in \{j + m, \dots, k - m\}$, by (4.2) we have that

$$d_{Y_l}(\partial Y_j, \partial Y_k) \geq E - C > G.$$

Thus Theorem 2.11 guarantees that there is a curve $\delta_l \in \eta$ disjoint from Y_l . Choose one such $\delta_l \in \eta$ for each $l \in [j + m, k - m]$. By the previous claim there are at most $4n$ integers $l' \in [j + m, k - m]$ such that $i(\delta_{l'}, \partial Y_{l'}) = 0$, and hence $l \mapsto \delta_l$ is at most $4n$ -to-1.

Therefore, η contains at least $\frac{k-j-2m+1}{4n} > \frac{k-j}{4n} - \frac{m}{2n}$ curves. It follows that

$$d_S(\partial Y_j, \partial Y_k) \geq \frac{k-j}{4n} - \left(\frac{m}{2n} + 1\right)$$

proving (4.3). This completes the proof of the theorem. \square

Theorem 4.3. Let $\{Y_k\}_{k=0}^\infty$ be an infinite sequence of subsurfaces satisfying conditions (1)-(4) in Theorem 4.1. Then there exists a unique $\nu \in \mathcal{EL}(S)$ so that any accumulation point of $\{\partial Y_k\}_{k=0}^\infty$ in $\mathcal{PML}(S)$ is supported on ν .

Proof. By Theorem 4.1, Inequality (4.3), the sequence $\{\partial Y_k\}_{k=0}^\infty$ is (multi-curve) quasi-geodesic in $\mathcal{C}(S)$. Furthermore $\mathcal{C}(S)$ is δ -hyperbolic. Thus the sequence converges to a point in the Gromov boundary of $\mathcal{C}(S)$. Theorem 2.3 completes the proof. \square

We complete this section by showing that \mathcal{P} is sufficient to imply the hypotheses of Theorem 4.1.

Proposition 4.4. *Any sequence $\{\gamma_k\}_{k=0}^\infty$ satisfying $\mathcal{P}(a)$ with $a > 2$ and $e_0 \geq E$ are the core curves of annuli satisfying conditions (1)–(4) of Theorem 4.1 with $n = m$. Consequently, $\{\gamma_k\}_{k=0}^\infty$ is a 1-Lipschitz, $(4m, \frac{3}{2})$ -quasi-geodesic in $\mathcal{C}(S)$ and there exists $\nu \in \mathcal{EL}(S)$ so that any accumulation point of $\{\gamma_k\}_{k=0}^\infty$ in $\mathcal{PML}(S)$ is supported on ν .*

Proof. Condition (i) of \mathcal{P} is the same as condition (1) of Theorem 4.1, while (ii) is just condition (4) with $n = m$. Condition (2) follows from Lemma 3.2. Finally, to see that condition (3) is satisfied, we note that $d_{\gamma_k}(\gamma_{k-m}, \gamma_{k+m}) \geq e_k > a^k E > 2E > E$, for all $k \geq m$. Furthermore, for $k-2m+1 \leq j \leq k-m$, $\gamma_j \pitchfork \gamma_k$ by Lemma 3.2, and similarly $\gamma_{j'} \pitchfork \gamma_k$, for $k+m \leq j' \leq k+2m-1$. For j and j' in these intervals, $i(\gamma_j, \gamma_{k-m}) = 0$ and $i(\gamma_{j'}, \gamma_{k+m}) = 0$. Therefore, by the triangle inequality, $d_{\gamma_k}(\gamma_j, \gamma_{j'}) \geq a^k E - 2 > E$, as required by (3). \square

4.1. Subsurface coefficient bounds. We will need estimates on all subsurface coefficients for a sequence satisfying \mathcal{P} . This follows from what we have done so far, together with similar arguments.

Proposition 4.5. *Given a sequence $\{\gamma_k\}_{k=0}^\infty$ satisfying $\mathcal{P}(a)$ with $a > 2$ and $e_0 \geq E$, then there exists $R > 0$ with the following property.*

(1) *If i, j, k satisfy $j \leq i - m$ and $i + m \leq k$ then $\gamma_i \pitchfork \gamma_k$, $\gamma_i \pitchfork \gamma_j$, and*

$$(4.5) \quad d_{\gamma_i}(\gamma_j, \gamma_k) \stackrel{\perp}{\succ}_R e_i \quad \text{and} \quad d_{\gamma_i}(\gamma_j, \nu) \stackrel{\perp}{\succ}_R e_i$$

(2) *If $W \subsetneq S$ is a proper subsurface, $W \neq \gamma_i$ for any i , then for any j, k with $\gamma_j \pitchfork W$ and $\gamma_k \pitchfork W$*

$$(4.6) \quad d_W(\gamma_j, \gamma_k) < R \quad \text{and} \quad d_W(\gamma_j, \nu) < R.$$

Let μ be a marking on S , then there is a constant $R(\mu)$ so that

For any k sufficiently large and $i \leq k - m$

$$(4.7) \quad d_{\gamma_i}(\mu, \gamma_k) \stackrel{\perp}{\succ}_{R(\mu)} e_i \quad \text{and} \quad d_{\gamma_i}(\mu, \nu) \stackrel{\perp}{\succ}_{R(\mu)} e_i$$

For any proper subsurface $W \neq \gamma_i$ for any i we have

$$(4.8) \quad d_W(\mu, \gamma_k) < R(\mu) \quad \text{and} \quad d_W(\mu, \nu) < R(\mu)$$

Proof. We begin with the proofs of (4.5) and (4.6). First note that since any accumulation point of $\{\gamma_k\}$ in $\mathcal{PML}(S)$ is supported on ν , any Hausdorff accumulation point of $\{\gamma_k\}$ contains ν . Thus, for any fixed, proper subsurface $W \subsetneq S$ and all sufficiently large k we have $\pi_W(\nu) \subseteq \pi_W(\gamma_k)$. Furthermore, since ν is an ending lamination, $\pi_W(\nu) \neq \emptyset$, and hence $d_W(\gamma_k, \nu) \leq 1$, for k sufficiently large. Therefore, for each of (4.5) and (4.6), the statement on the left implies the one on the right after increasing the constant by at most 1. Thus it suffices to prove the two statements on the left.

We begin with (4.5). From the conditions in \mathcal{P} , we have $d_{\gamma_i}(\gamma_{i-m}, \gamma_{i+m}) = e_i$. By Theorem 4.1 (which is applicable according to Proposition 4.4), $\{\gamma_l\}_{l=i+m}^k$ is a 1-Lipschitz $(4m, 3/2)$ -quasi-geodesic such that every curve

has nonempty projection to γ_i . Therefore, by Theorem 2.11 and the triangle inequality we have

$$|d_{\gamma_i}(\gamma_{i-m}, \gamma_k) - d_{\gamma_i}(\gamma_{i-m}, \gamma_{i+m})| \leq d_{\gamma_i}(\gamma_{i+m}, \gamma_k) \leq G.$$

Note that G depends only on m . Similar reasoning implies

$$|d_{\gamma_i}(\gamma_j, \gamma_k) - d_{\gamma_i}(\gamma_{i-m}, \gamma_k)| \leq d_{\gamma_i}(\gamma_j, \gamma_{i-m}) \leq G.$$

Combining these we have

$$\begin{aligned} |d_{\gamma_i}(\gamma_j, \gamma_k) - d_{\gamma_i}(\gamma_{i-m}, \gamma_{i+m})| &= |d_{\gamma_i}(\gamma_j, \gamma_k) - d_{\gamma_i}(\gamma_{i-m}, \gamma_k) \\ &\quad + d_{\gamma_i}(\gamma_{i-m}, \gamma_k) - d_{\gamma_i}(\gamma_{i-m}, \gamma_{i+m})| \\ &\leq 2G \end{aligned}$$

It follows that $d_{\gamma_i}(\gamma_j, \gamma_k) \stackrel{\pm}{\asymp}_{2G} e_i$. For $R \geq 2G$, (4.5) holds.

We now move on to (4.6), and without loss of generality assume $j \leq k$. If $k \leq j + 2m - 1$, then the conditions in \mathcal{P} together with Lemma 3.2 imply $i(\gamma_j, \gamma_k) \leq b_2$, so by Lemma 2.5, $d_W(\gamma_j, \gamma_k) \leq 2b_2 + 1$.

Next, suppose that $k = j + 2m$. Let γ'_k be the element guaranteed by \mathcal{P} , so that $\gamma_k = \mathcal{D}_{\gamma_{k-m}}^{e_{k-m}}(\gamma'_k)$. There are two cases to consider depending on whether $\gamma'_k \not\cap W$ or $\gamma'_k \cap W$. If $\gamma'_k \not\cap W$, then since $\gamma_k = \mathcal{D}_{\gamma_{k-m}}^{e_{k-m}}(\gamma'_k) \cap W$, we must have $\gamma_{k-m} \cap W$. Now observe that $j \leq k - m = j + m \leq j + 2m - 1$ and $k - m \leq k \leq k - m + 2m - 1$. It follows from the previous paragraph that $d_W(\gamma_j, \gamma_{k-m}) \leq 2b_2 + 1$ and $d_W(\gamma_{k-m}, \gamma_k) \leq 2b_2 + 1$, hence

$$d_W(\gamma_j, \gamma_k) \leq 4b_2 + 2.$$

Now suppose $\gamma'_k \cap W$. If $\gamma_{k-m} \cap W$ then just as in the first case we have $d_W(\gamma_j, \gamma_k) \leq 4b_2 + 2$. Suppose then that $\gamma_{k-m} \not\cap W$. If W is not an annulus, then $\pi_W(\gamma_k) = \pi_W(\gamma'_k)$ since $\mathcal{D}_{\gamma_{k-m}}$ is supported outside W . Therefore

$$d_W(\gamma_j, \gamma_k) = d_W(\gamma_j, \gamma'_k) \leq 2b_2 + 1$$

since $i(\gamma_j, \gamma'_k) \leq b_2$. If W is an annulus, because $W \neq \gamma_{k-m}$ and $\gamma_{k-m} \not\cap W$, it easily follows that

$$d_W(\gamma_j, \gamma_k) \leq d_W(\gamma_j, \gamma'_k) + d_W(\gamma'_k, \gamma_k) \leq (2b_2 + 1) + 1$$

(see e.g. [FLM01]). Therefore, we have shown that if $k \leq j + 2m$, we have

$$(4.9) \quad d_W(\gamma_j, \gamma_k) \leq 4b_2 + 2.$$

Now we suppose $k > j + 2m$. Setting $\delta = \partial W$, as in the proof of Theorem 4.1 we let $\mathcal{I}(\delta) = \{s \in [j, k] \mid i(\delta, \gamma_s) \neq 0\}$. Similarly, we let $\mathcal{I}(W) = \{s \in [j, k] \mid \gamma_s \cap W\}$, and observe that $\mathcal{I}(\delta) \subseteq \mathcal{I}(W)$.

Note that $j, k \in \mathcal{I}(W)$, and we let $s \leq r$ be such that $[j, s], [r, k] \subseteq \mathcal{I}(W)$ are maximal subintervals of $\mathcal{I}(W)$ containing j and k , respectively (if $\mathcal{I}(W) = [j, k]$, we can arbitrarily choose $j \leq s < k$ and $r = s + 1$ for the argument below). By Claim 4.2, $r - s \leq 4m$.

Note that since any $2m$ consecutive curves fill S , either $r - s \leq 2m$, or else there exists $s', r' \in \mathcal{I}(W)$ such that $s < s' \leq r' < r$ and $r - r', r' - s', s' - s \leq 2m$. For example, consider the extremal case that $r - s = 4m$. Then

$$s' = \max \mathcal{I}(W) \cap [s, s + 2m] \quad \text{and} \quad r' = \min \mathcal{I}(W) \cap [s + 2m, r]$$

have the desired properties. The general case is similar. By the triangle inequality and (4.9) we have

$$(4.10) \quad d_W(\gamma_s, \gamma_r) \leq d_W(\gamma_s, \gamma_{s'}) + d_W(\gamma_{s'}, \gamma_{r'}) + d_W(\gamma_{r'}, \gamma_r) \leq 12b_2 + 6.$$

Since $\{\gamma_l\}_{l=j}^s$ and $\{\gamma_l\}_{l=r'}^k$ are 1-Lipschitz $(4m, 3/2)$ -quasi-geodesics with $\gamma_l \pitchfork W$ for all $l \in [j, s] \cup [r', k]$, we can apply Theorem 2.11, and so the triangle inequality and (4.10) gives us

$$d_W(\gamma_j, \gamma_k) \leq d_W(\gamma_j, \gamma_s) + d_W(\gamma_s, \gamma_r) + d_W(\gamma_r, \gamma_k) \leq 2G + 12b_2 + 6.$$

So the inequality on the left of (4.6) holds for any $R \geq 2G + 12b_2 + 6$. This completes the proof of the first four estimates.

Given a marking μ , note that the intersection number of any curve in μ and any of the curves in the set of filling curves $\gamma_0, \dots, \gamma_{2m-1}$ is bounded. Then the estimates in (4.7) follow from the ones in (4.5) and Lemma 2.5 respectively. Similarly the estimates in (4.6) follow from the ones in (4.8). \square

5. MEASURES SUPPORTED ON LAMINATIONS

In this section we begin by proving intersection number estimates for a sequence of curves satisfying \mathcal{P} . Using these estimates, we decompose the sequence into m subsequences and prove that these converge in $\mathcal{PML}(S)$. In the next section, we will show that these m limits are precisely the vertices of the simplex of measures on the single topological lamination ν from Proposition 4.4.

5.1. Intersection number estimates. Here we estimate the intersection numbers of curves in the sequence of curves $\{\gamma_k\}_{k=0}^\infty$ satisfying \mathcal{P} . The estimates will be in terms of the constant b and sequence $\{e_k\}$ fixed above. Specifically, given $i, k \in \mathbb{N}$ with $k \geq i$, define

$$(5.1) \quad A(i, k) := \prod_{\substack{i+m \leq j < k \text{ and} \\ j \equiv k \pmod{m}}} be_j.$$

When the set of indices of the product is the empty set we define the product to be 1. It is useful to observe that for $k \geq i + 2m$,

$$A(i, k) = be_{k-m} A(i, k - m).$$

It is also useful to arrange the indices as in (3.2) in the following form

$$(5.2) \quad \begin{array}{cccc} i & i + 1 & \cdots & i + m - 1 \\ i + m & i + m + 1 & \cdots & i + 2m - 1 \\ i + 2m & i + 2m + 1 & \cdots & \end{array}$$

Then $A(i, k)$ is 1 exactly when k is in the first or second row. If k is below these rows, then the product defining $A(i, k)$ is over all indices j directly above k , up to and including the entry in the second row.

We now state the main estimate on intersection numbers.

Theorem 5.1. *Suppose $\{\gamma_k\}_{k=0}^\infty$ is a sequence on a surface S satisfying $\mathcal{P}(a)$. For a is sufficiently large, there is a constant $\kappa = \kappa(a) > 1$, so that for each i, k with $k \geq i + m$ we have*

$$(5.3) \quad i(\gamma_i, \gamma_k) \stackrel{*}{\asymp}_\kappa A(i, k).$$

Recall that for $i \leq k < i + m$, $i(\gamma_i, \gamma_k) = 0$. Combining this with the theorem gives estimates on all intersection numbers $i(\gamma_i, \gamma_k)$, up to a uniform multiplicative error.

Throughout all that follows, we will assume that a sequence of curves $\{\gamma_k\}_{k=0}^\infty$ satisfies $\mathcal{P} = \mathcal{P}(a)$ for $a > 1$.

Outline of the proof: The proof is rather complicated involving multiple induction arguments, so we sketch the approach before diving into the details. The upper bound on $i(\gamma_i, \gamma_k)$ is proved first, and is valid for any $a > 1$. We start by recursively defining a function $K(i, k)$ for all nonnegative integers $i \leq k$. By induction, we will prove that

$$i(\gamma_i, \gamma_k) \leq K(i, k)A(i, k).$$

By a second induction, we will bound $K(i, k) \leq K_1 = K_1(a)$, with the bound $K_1(a)$ a decreasing function of a . Next, we will recursively define a function $K'(i, k) = K'(i, k, a)$. By another induction, we prove that

$$i(\gamma_i, \gamma_k) \geq K'(i, k)A(i, k).$$

For a sufficiently large, we prove that $K'(i, k, a) \geq K_2 = K_2(a) > 0$. Setting $\kappa = \max\{K_1, \frac{1}{K_2}\}$ will prove the theorem.

Upper bound. Recall from \mathcal{P} (Definition 3.1) that for any $k \geq 2m$, the set of curves $\{\gamma_l\}_{l=k-2m}^{k-1}$ fill the surface, and the curve γ'_k intersects each of these at most b_2 times. Consequently, all complementary components of $S \setminus (\gamma_{k-2m} \cup \dots \cup \gamma_{k-1})$ are either disks or once-punctured disks containing at most $2mb_2$ pairwise disjoint arcs of γ'_k . In examples we may have many fewer than $2mb_2$ such arcs, and it is useful to keep track of this constant on its own. Consequently, we set

$$(5.4) \quad B \leq 2mb_2$$

to be the maximum number of arcs in any complementary component (over all configurations in minimal position).

We are now ready for a recursive definition which will be used in the bounds on intersection numbers (it is useful again to picture the indices as

in (5.2)):

$$K(i, k) = \begin{cases} 0 & \text{for } i \leq k < i + m \\ b_2 & \text{for } i + m \leq k < i + 2m \\ K(i, k - m) + 2B \sum_{l=k-2m}^{k-1} \frac{A(i, l)}{A(i, k)} K(i, l) & \text{for } i + 2m \leq k \end{cases}$$

Lemma 5.2. *For all $i \leq k$, we have $i(\gamma_i, \gamma_k) \leq K(i, k)A(i, k)$.*

The proof takes advantage of the following well-known estimate on the intersection of two curves after applying a power of a Dehn twist on one proved in the Appendix A of [FLP79, Exposé 4], see also Lemma 4.2 in [Iva92, §4].

Proposition 5.3. (INTERSECTION NUMBER AFTER DEHN TWIST)

Let δ, δ' , and β be curves in $\mathcal{C}(S)$. Then for any integer e

$$(5.5) \quad \left| i(\mathcal{D}_\beta^e \delta, \delta') - |e| i(\beta, \delta) i(\beta, \delta') \right| \leq i(\delta, \delta').$$

As above, \mathcal{D}_β is a Dehn twist in β . This proposition has the following general application to intersection numbers of curves with the curves in our sequence.

Proposition 5.4. *For any curve δ and any $k \geq 2m$, we have*

$$|i(\delta, \gamma_k) - b e_{k-m} i(\delta, \gamma_{k-m})| \leq 2B \sum_{l=k-2m}^{k-1} i(\delta, \gamma_l).$$

Proof. Since $\gamma_k = \mathcal{D}_{\gamma_{k-m}}^{e_{k-m}}(\gamma'_k)$, Proposition 5.3 implies

$$|i(\delta, \gamma_k) - b e_{k-m} i(\delta, \gamma_{k-m})| \leq i(\delta, \gamma'_k).$$

Assume all curves intersect minimally transversely and that there are no triple points of intersection. From the definition of B , all complementary components of $S \setminus (\gamma_{k-2m} \cup \dots \cup \gamma_{k-1})$ contain at most B pairwise disjoint arcs of γ'_k . Therefore, between any two consecutive intersection points of δ with $\gamma_{k-2m} \cup \dots \cup \gamma_{k-1}$, there are at most $2B$ intersection points with γ'_k (any two arcs in a disk component can intersect at most once, and in a once-punctured disk component can intersect in at most two points). Therefore,

$$i(\delta, \gamma'_k) \leq 2B \sum_{l=k-2m}^{k-1} i(\delta, \gamma_l).$$

Combining this with the above inequality proves the proposition. \square

We are now ready for the

Proof of Lemma 5.2. Fix i . The proof is by induction on k . For $i \leq k < i + m$, $i(\gamma_i, \gamma_k) = 0$, $K(i, k) = 0$ and $A(i, k) = 1$, so the lemma follows. Similarly, for $i + m \leq k < i + 2m$, $i(\gamma_i, \gamma_k) \leq b_2$, $K(i, k) = b_2$, and $A(i, k) = 1$, so again the theorem follows. Now suppose that $k \geq i + 2m$, and assuming

that $i(\gamma_i, \gamma_l) \leq K(i, l)A(i, l)$ for all $i \leq l < k$, we must prove $i(\gamma_i, \gamma_k) \leq K(i, k)A(i, k)$.

Applying Proposition 5.4 to the case $\delta = \gamma_i$, we have

$$|i(\gamma_i, \gamma_k) - be_{k-m}i(\gamma_i, \gamma_{k-m})| \leq 2B \sum_{l=k-2m}^{k-1} i(\gamma_i, \gamma_l).$$

Therefore, we have

$$i(\gamma_i, \gamma_k) \leq be_{k-m}i(\gamma_i, \gamma_{k-m}) + 2B \sum_{l=k-2m}^{k-1} i(\gamma_i, \gamma_l).$$

Applying the inductive hypothesis and the definitions of A and K to this inequality we obtain

$$\begin{aligned} i(\gamma_i, \gamma_k) &\leq be_{k-m}i(\gamma_i, \gamma_{k-m}) + 2B \sum_{l=k-2m}^{k-1} i(\gamma_i, \gamma_l) \\ &\leq be_{k-m}K(i, k-m)A(i, k-m) + 2B \sum_{l=k-2m}^{k-1} K(i, l)A(i, l) \\ &= A(i, k)K(i, k-m) + A(i, k)2B \sum_{l=k-2m}^{k-1} \frac{A(i, l)}{A(i, k)} K(i, l) \\ &= A(i, k) \left(K(i, k-m) + 2B \sum_{l=k-2m}^{k-1} \frac{A(i, l)}{A(i, k)} K(i, l) \right) \\ &= A(i, k)K(i, k), \end{aligned}$$

as required. \square

Next we prove that $K(i, k)$ is uniformly bounded, and in particular:

Proposition 5.5. *There exists $K_1 = K_1(a) > 0$ so that for all $i \leq k$, $K(i, k) \leq K_1$ and in particular, $i(\gamma_i, \gamma_k) \leq K_1A(i, k)$. As a function of a , $K_1(a)$ is decreasing.*

For the proof of this proposition, we will need the following bound.

Lemma 5.6. *For all $i \leq l < k$, we have*

$$\frac{A(i, l)}{A(i, k)} \leq a^{1 - \lfloor \frac{k-i}{m} \rfloor}.$$

Proof. If $k < i + 2m$, then $A(i, l), A(i, k) = 1$ and $a^{1 - \lfloor \frac{k-i}{m} \rfloor} \geq 1$, so the inequality follows.

Now assume $k \geq i + 2m$. By definition, we have

$$\frac{A(i, l)}{A(i, k)} = \frac{\prod_{\substack{i+m \leq j' < l \text{ and} \\ j' \equiv l \pmod{m}}} b e_{j'}}{\prod_{\substack{i+m \leq j < k \text{ and} \\ j \equiv k \pmod{m}}} b e_j}$$

(where $A(i, l)$ is 1 if $l < i + 2m$). Observe that the denominator has $r = \lfloor \frac{k-(i+m)}{m} \rfloor = \lfloor \frac{k-i}{m} \rfloor - 1 > 0$ terms in the product, indexed by $j \in \{k-m, k-2m, \dots, k-rm\}$, while the numerator has $s = \max\{0, \lfloor \frac{l-i}{m} \rfloor - 1\} \geq 0$ terms, indexed by $j' \in \{l-m, l-2m, \dots, l-sm\}$ (possibly the empty set). Since $l < k$, $s \leq r$. Moreover, we have $k-pm > l-pm$, and thus $e_{k-pm} > a e_{l-pm}$ by (3.1), for all $p = 1, \dots, s$. Since (3.1) also implies $e_j > a$ for all $j \geq 1$, combining these bounds with the equation above gives

$$\frac{A(i, l)}{A(i, k)} = \prod_{p=1}^s \frac{e_{l-pm}}{e_{k-pm}} \prod_{p=s+1}^r \frac{1}{e_{k-pm}} < \prod_{p=1}^s a^{-1} \prod_{p=s+1}^r a^{-1} = a^{-r} = a^{1-\lfloor \frac{k-i}{m} \rfloor},$$

as required. \square

As an application, of Lemma 5.6, we prove

Lemma 5.7. *For all $i \leq k$ we have*

$$K(i, k) \leq b_2 \prod_{i+m \leq j < k} (1 + 4mBa^{1-\lfloor \frac{j-i+1}{m} \rfloor}).$$

As above, the empty product is declared to be 1.

Proof. The proof is by induction on k . Since $K(i, k) \leq b_2$ for $i \leq k < i + 2m$, the lemma clearly holds for all such k . Now assume that $k \geq i + 2m$, and assume that the lemma holds for all integers less than k and at least i . Let l_0 be such that $k - 2m \leq l_0 \leq k - 1$ and

$$K(i, l_0) = \max\{K(i, l) \mid k - 2m \leq l \leq k - 1\}.$$

From this, the definition of $K(i, k)$, and from Lemma 5.6 we have

$$\begin{aligned} K(i, k) &= K(i, k-m) + 2B \sum_{l=k-2m}^{k-1} \frac{A(i, l)}{A(i, k)} K(i, l) \\ &\leq K(i, l_0) (1 + 2B \sum_{l=k-2m}^{k-1} a^{1-\lfloor \frac{k-i}{m} \rfloor}) \\ &= K(i, l_0) (1 + 4mBa^{1-\lfloor \frac{k-i}{m} \rfloor}) \end{aligned}$$

Since $l_0 < k$, the proposed bound on $K(i, l_0)$ holds by the inductive assumption. Next, observe that the proposed upper bound is an increasing function of k . Indeed, the required bound for $K(i, k)$ is obtained from the one for

$K(i, k-1)$ by multiplying by a number greater than or equal to 1. By this monotonicity, the above bound implies

$$\begin{aligned} K(i, k) &\leq K(i, l_0)(1 + 4mBa^{1-\lfloor \frac{k-i}{m} \rfloor}) \\ &\leq \left(b_2 \prod_{i+m \leq j < k-1} (1 + 4mBa^{1-\lfloor \frac{j-i+1}{m} \rfloor}) \right) (1 + 4mBa^{1-\lfloor \frac{k-i}{m} \rfloor}) \\ &= b_2 \prod_{i+m \leq j < k} (1 + 4mBa^{1-\lfloor \frac{j-i+1}{m} \rfloor}). \end{aligned}$$

This completes the proof. \square

We are now ready for the

Proof of Proposition 5.5. The upper bound on $K(i, k)$ in Lemma 5.7 is itself bounded above by the infinite product

$$K_1(a) = b_2 \prod_{j=i+m}^{\infty} (1 + 4mBa^{1-\lfloor \frac{j-i+1}{m} \rfloor}) = b_2 \prod_{l=0}^{\infty} (1 + 4mBa^{-\lfloor \frac{l+1}{m} \rfloor}).$$

We will be done if we prove that this product is convergent, for all $a > 1$, since the product then clearly defines a decreasing function of a .

The infinite product converges if and only if the infinite series obtained by taking logarithms does. Since $\log(1+x) \leq x$ we have

$$\begin{aligned} \log \left(b_2 \prod_{l=0}^{\infty} (1 + 4mBa^{-\lfloor \frac{l+1}{m} \rfloor}) \right) &= \log(b_2) + \sum_{l=0}^{\infty} \log(1 + 4mBa^{-\lfloor \frac{l+1}{m} \rfloor}) \\ &\leq \log(b_2) + 4mB \sum_{l=0}^{\infty} a^{-\lfloor \frac{l+1}{m} \rfloor} \end{aligned}$$

The last expression is essentially a geometric series, and hence converges for all $a > 1$, completing the proof. \square

Lower bound. Let b_1 be the constant in \mathcal{P} (Definition 3.1). We assume $a > 1$ is sufficiently large so that

$$(5.6) \quad C = 8mBK_1 \sum_{j=1}^{\infty} a^{-j} < b_1.$$

(which is possible since $K_1 = K_1(a)$ is decreasing by Proposition 5.5). For all $k \geq i+m$, define the function $K'(i, k)$ by the recursive formula for all $k \geq i+m$

$$K'(i, k) = \begin{cases} C & \text{for } i+m \leq k < i+2m \\ K'(i, k-m) - 2B \sum_{l=k-2m}^{k-1} \frac{A(i, l)}{A(i, k)} K(i, l) & \text{for } i+2m \leq k \end{cases}$$

Lemma 5.8. For all $k \geq i+m$, we have $i(\gamma_i, \gamma_k) \geq K'(i, k)A(i, k)$.

Proof. Fix integer $i \geq 0$. The proof is by induction on k . For the base case, we let $i + m \leq k < i + 2m$. Then $A(i, k) = 1$ and $K'(i, k) = C < b_1$, while $i(\gamma_i, \gamma_k) \geq b_1$, and hence $i(\gamma_i, \gamma_k) \geq K'(i, k)A(i, k)$. We assume therefore that $k \geq i + 2m$ and that the lemma is true for all $i + m \leq l < k$.

Applying Proposition 5.4 to the curve $\delta = \gamma_i$, together with Lemma 5.2 and the inductive hypothesis we have

$$\begin{aligned}
i(\gamma_i, \gamma_k) &\geq e_{k-m} b i(\gamma_i, \gamma_{k-m}) - 2B \sum_{l=k-2m}^{k-1} i(\gamma_i, \gamma_l) \\
&\geq e_{k-m} b K'(i, k-m) A(i, k-m) - 2B \sum_{l=k-2m}^{k-1} K(i, l) A(i, l) \\
&= A(i, k) \left(K'(i, k-m) - 2B \sum_{l=k-2m}^{k-1} \frac{A(i, l)}{A(i, k)} K(i, l) \right) \\
&= A(i, k) K'(i, k)
\end{aligned}$$

as required. \square

Lemma 5.9. *Setting $K_2 = C/2 > 0$, then whenever $k \geq i + m$, $K'(i, k) \geq K_2$.*

Proof. If $i + m \leq k < i + 2m$, then $K'(i, k) = C > C/2 = K_2 > 0$. Suppose now that $k \geq i + 2m$, and let $k = p + sm$, where s and p are positive integers with $i + m \leq p < i + 2m$ and $p \equiv k \pmod{m}$. Note that

$$\lfloor \frac{k-i}{m} \rfloor = \lfloor \frac{p+sm-i}{m} \rfloor = s + \lfloor \frac{p-i}{m} \rfloor = s + 1.$$

By Lemma 5.6, it follows that for all $l < k$, we have $\frac{A(i, l)}{A(i, k)} \leq a^{-s}$. Then from the definition of K' and Proposition 5.5 we have

$$\begin{aligned}
K'(i, k) &= K'(i, k-m) - 2B \sum_{l=k-2m}^{k-1} \frac{A(i, l)}{A(i, k)} K(i, l) \\
&\geq K'(i, k-m) - 2B \sum_{l=k-2m}^{k-1} a^{-s} K_1 \\
&\geq K'(i, k-m) - 2B(2m)a^{-s} K_1 = K'(i, k-m) - 4mBK_1 a^{-s}
\end{aligned}$$

Iterating this inequality s times implies

$$K'(i, k) \geq K'(i, p) - 4mBK_1 \sum_{q=1}^s a^{-q}.$$

Since $i + m \leq p < i + 2m$, $K'(i, p) = C = 8mBK_1 \sum_{j=1}^{\infty} a^{-j}$ and hence

$$K'(i, k) \geq 4mBK_1 \left(2 \sum_{j=1}^{\infty} a^{-j} - \sum_{q=1}^s a^{-q} \right) \geq 4mBK_1 \sum_{j=1}^{\infty} a^{-j} = C/2 = K_2.$$

This completes the proof. \square

Proof of Theorem 5.1. For $a > 1$ satisfying (5.6), we have proved that for all $k \geq i + m$,

$$K_2 A(i, k) \leq i(\gamma_i, \gamma_k) \leq K_1 A(i, k).$$

Since $K_1, K_2 > 0$, setting $\kappa = \max\{K_1, \frac{1}{K_2}\}$ finishes the proof. \square

Convention. From this point forward, we will assume that $\mathcal{P} = \mathcal{P}(a)$ always has $a > 1$ sufficiently large so that (5.6) is satisfied, and consequently the intersection numbers of curves in any sequence $\{\gamma_k\}_{k=0}^{\infty}$ satisfies (5.3) in Theorem 5.1. For concreteness, we note that from (5.6), $a \geq 16 > 2$ (though in fact, it is much larger).

5.2. Convergence in $\mathcal{ML}(S)$. Consider again a sequence of curves $\{\gamma_k\}_{k=0}^{\infty}$ which satisfies the conditions of Theorem 5.1. Let $\nu \in \mathcal{EL}(S)$ be the lamination from Proposition 4.4. In this section we will prove this sequence naturally splits into m convergent subsequences in $\mathcal{PML}(S)$.

For each $h = 0, \dots, m - 1$ and $i \in \mathbb{N}$ let

$$(5.7) \quad c_i^h = A(0, im + h) = \prod_{j=1}^{i-1} b_{e_{jm+h}}.$$

where A is defined in (5.1).

For each $h = 0, 1, \dots, m - 1$, define the subsequence γ_i^h of the sequence $\{\gamma_k\}_{k=0}^{\infty}$ by

$$(5.8) \quad \gamma_i^h = \gamma_{im+h}.$$

The main result of this section is the following theorem.

Theorem 5.10. *Suppose $\{\gamma_k\}_{k=0}^{\infty}$ satisfies \mathcal{P} . Then for each $h = 0, 1, \dots, m - 1$, there exists a transverse measure $\bar{\nu}^h$ on ν so that*

$$\lim_{i \rightarrow \infty} \frac{\gamma_i^h}{c_i^h} = \bar{\nu}^h$$

in $\mathcal{ML}(S)$, where γ_i^h and c_i^h are as above.

We will need the following generalization of Theorem 5.1.

Lemma 5.11. *For any curve δ , there exists $\kappa(\delta) > 0$ and $N(\delta) > 0$ so that for all $k \geq N(\delta)$*

$$(5.9) \quad i(\delta, \gamma_k) \stackrel{*}{\asymp}_{\kappa(\delta)} A(0, k).$$

Remark 5.12. Note that in Theorem 5.1, we estimate $i(\gamma_i, \gamma_k)$ with a uniform multiplicative constant κ that works for any two curves γ_i and γ_k , but the comparison is with $A(i, k)$ rather than $A(0, k)$. On the other hand, the ratio of $A(0, k)$ and $A(i, k)$ is bounded by a constant depending on i , and not k , so the lemma for $\delta = \gamma_i$ is an immediate consequence of that theorem.

Proof. First we note that by Theorem 5.1, we have

$$i(\gamma_i, \gamma_k) \stackrel{*}{\succ}_{\kappa} A(i, k).$$

From the definition of A , and the fact that $\{e_j\}_{j=0}^{\infty}$ is an increasing sequence, it follows that for each $i = 0, \dots, 2m-1$, and all $k \geq i$, we have the bound

$$1 \leq \frac{A(0, k)}{A(i, k)} \leq b^2 e_{2m} e_{3m}.$$

Setting $\kappa_0 = \kappa b^2 e_{2m} e_{3m}$, for each $i = 0, \dots, 2m-1$, we have

$$(5.10) \quad i(\gamma_i, \gamma_k) \stackrel{*}{\succ}_{\kappa_0} A(0, k).$$

Next, let $d = 2m\kappa_0$. Note that since $\gamma_0, \dots, \gamma_{2m-1}$ fills S , the set of measured laminations

$$\Delta = \{\bar{\lambda} \mid \sum_{j=0}^{2m-1} i(\gamma_j, \bar{\lambda}) \stackrel{*}{\succ}_d 1\} \subset \mathcal{ML}(S)$$

is compact. From (5.10) we have $\{\frac{\gamma_k}{A(0, k)}\}_{k=3m}^{\infty} \subset \Delta$.

Let $\nu \in \mathcal{EL}(S)$ be the lamination from Proposition 4.4. Since ν is an ending lamination, the set of measures $\bar{\nu} \in \Delta$ supported on ν is a compact subset. By continuity of the intersection number i , there exists $c(\delta) > 0$ so that $i(\delta, \bar{\nu}) \stackrel{*}{\succ}_{c(\delta)} 1$ for all such $\bar{\nu}$.

Let $K(\delta) \subset \mathcal{ML}(S)$ be a compact neighborhood which contains the set of measures $\bar{\nu}$ which are supported on ν and are in Δ . By continuity of i again, we can take $K(\delta)$ sufficiently small so that there exists $\kappa(\delta) > 0$ such that $i(\delta, \bar{\lambda}) \stackrel{*}{\succ}_{\kappa(\delta)} 1$ for all $\bar{\lambda} \in K(\delta)$. Since every accumulation point of $\{\frac{\gamma_k}{A(0, k)}\}_{k=3m}^{\infty}$ is a measure $\bar{\nu} \in \Delta$ supported on ν , it follows that there exists $N(\delta)$ so that for all $k \geq N(\delta)$, $\frac{\gamma_k}{A(0, k)} \in K(\delta)$. Consequently, for all $k \geq N(\delta)$, we have $i(\delta, \gamma_k) \stackrel{*}{\succ}_{\kappa(\delta)} A(0, k)$, which completes the proof. \square

Using the estimates from Lemma 5.11, we prove the next lemma. Theorem 5.10 will then follow easily.

Lemma 5.13. *For any curve δ and any $h = 0, \dots, m-1$, the sequence $\{i(\delta, \frac{\gamma_i^h}{c_i^h})\}_{i=0}^{\infty}$ converges.*

Proof. By Proposition 5.4 we have that

$$\left| i(\delta, \gamma_{im+h}) - e_{(i-1)m+h} b i(\delta, \gamma_{(i-1)m+h}) \right| \leq 2B \sum_{l=(i-2)m+h}^{im+h-1} i(\delta, \gamma_l).$$

Dividing both sides by $c_i^h = A(0, im + h) = be_{(i-1)m+h}A(0, (i-1)m + h)$, and letting $\kappa(\delta)$ be the constant from Lemma 5.11, it follows that for all $h = 0, \dots, m-1$, and i sufficiently large

$$\begin{aligned} \left| i(\delta, \frac{\gamma_{im+h}}{c_i^h}) - i(\delta, \frac{\gamma_{(i-1)m+h}}{c_{i-1}^h}) \right| &\leq \frac{2B}{A(0, im + h)} \left(\sum_{l=(i-2)m+h}^{im+h-1} i(\delta, \gamma_l) \right) \\ &\leq \frac{2B}{A(0, im + h)} \left(\sum_{l=(i-2)m+h}^{im+h-1} \kappa(\delta)A(0, l) \right) \\ &= \sum_{l=(i-2)m+h}^{im+h-1} 2B\kappa(\delta) \frac{A(0, l)}{A(0, im + h)} \end{aligned}$$

Lemma 5.6 implies that the expressions in the final sum admit the following bounds:

$$\frac{A(0, l)}{A(0, im + h)} \leq a^{1-\lfloor \frac{im+h-l}{m} \rfloor} = a^{1-i}$$

Since $\gamma_i^h = \gamma_{im+h}$, we have

$$\left| i(\delta, \frac{\gamma_i^h}{c_i^h}) - i(\delta, \frac{\gamma_{i-1}^h}{c_{i-1}^h}) \right| \leq 4mB\kappa(\delta)a^{1-i}.$$

Consequently, for all $i > j$ sufficiently large, applying this inequality and the triangle inequality we have

$$\left| i(\delta, \frac{\gamma_i^h}{c_i^h}) - i(\delta, \frac{\gamma_j^h}{c_j^h}) \right| \leq 4mB\kappa(\delta) \sum_{l=j+1}^i a^{1-l}.$$

By taking i and j sufficiently large, the (partial) sum of the geometric series on the right can be made arbitrarily small. In particular, $\{i(\delta, \frac{\gamma_i^h}{c_i^h})\}$ is a Cauchy sequence, hence converges. \square

Proof of Theorem 5.10. Fix $h \in \{0, \dots, m-1\}$. Since the intersection numbers $\{i(\delta, \frac{\gamma_i^h}{c_i^h})\}_{i=0}^\infty$ converge for all simple closed curves δ , it follows that $\{\frac{\gamma_i^h}{c_i^h}\}_{i=0}^\infty$ converges to some lamination $\bar{\nu}^h$ in $\mathcal{ML}(S)$ (since $\mathcal{ML}(S)$ is a closed subset of $\mathbb{R}^{\mathcal{C}(S)}$). By Proposition 4.4, $\bar{\nu}^h$ is supported on ν . \square

6. ERGODIC MEASURES

We continue to assume throughout the rest of this section that $\{\gamma_k\}_{k=0}^\infty$ satisfies \mathcal{P} and that $\{\frac{\gamma_i^h}{c_i^h}\}_{i=0}^\infty$ for $h = 0, \dots, m$ are the subsequences defined in the previous section limiting to $\bar{\nu}^h$ supported on ν by Theorem 5.10 for each $h = 0, \dots, m-1$. We say that $\bar{\nu}^h$ and $\bar{\nu}^{h'}$ are *not absolutely continuous* if neither is absolutely continuous with respect to the other one. Note that this is weaker than requiring that the measures be mutually singular.

The measures supported on ν form a cone on a simplex which we refer to as *the simplex of measures on ν* . The vertices of the simplex are the *ergodic measures*, and are well-defined up to multiplication by positive scalars. We denote (choices of) the ergodic measures by $\bar{\mu}^0, \dots, \bar{\mu}^{d-1}$ where $0 \leq d \leq \xi(S)$ is the dimension of the cone on the simplex.

The ergodic measures are mutually singular since the generic points are disjoint. It follows that if we write $\bar{\nu}^h$ and $\bar{\nu}^{h'}$ as nonnegative linear combinations of $\bar{\mu}^0, \dots, \bar{\mu}^{d-1}$, then $\bar{\nu}^h$ and $\bar{\nu}^{h'}$ are not absolutely continuous if and only if there exists $\bar{\mu}^j, \bar{\mu}^{j'}$ so that $\bar{\mu}^j$ has positive coefficient for $\bar{\nu}^h$ and zero coefficient for $\bar{\nu}^{h'}$, while $\bar{\mu}^{j'}$ has positive coefficient for $\bar{\nu}^{h'}$ and zero coefficient for $\bar{\nu}^h$.

The aim of this section is to show that $d = m$, and in particular, ν is nonuniquely ergodic. In fact, we will prove that up to scaling and reindexing we have $\bar{\mu}^h = \bar{\nu}^h$.

Using the estimates on the intersection numbers from Theorem 5.1 we first show that the measures $\bar{\nu}^h$ for $h = 0, \dots, m-1$, are pairwise not absolutely continuous.

Theorem 6.1. *Let $h, h' \in \{0, \dots, m-1\}$ and $h \neq h'$. Then*

$$\lim_{i \rightarrow \infty} \frac{i(\gamma_i^h, \bar{\nu}^h)}{i(\gamma_i^h, \bar{\nu}^{h'})} = \infty \quad \text{and} \quad \lim_{i \rightarrow \infty} \frac{i(\gamma_i^{h'}, \bar{\nu}^{h'})}{i(\gamma_i^{h'}, \bar{\nu}^h)} = \infty.$$

In particular, the measures $\bar{\nu}^h$ and $\bar{\nu}^{h'}$ are not absolutely continuous with respect to each other.

The last statement is a consequence of the two limits, for if $\bar{\nu}^h$ and $\bar{\nu}^{h'}$ were positive linear combinations of the same set of ergodic measures, then these ratios would have to be bounded.

Proof. For $h \neq h'$, we will calculate that

$$(6.1) \quad i(\gamma_0^h, \gamma_{i+1}^h) i(\gamma_i^h, \bar{\nu}^h) \stackrel{*}{\asymp} 1 \quad \text{and} \quad \lim_{i \rightarrow \infty} i(\gamma_0^h, \gamma_{i+1}^h) i(\gamma_i^h, \bar{\nu}^{h'}) = 0.$$

Dividing the first equation by the second and taking limit (and doing the same with the roles of h and h' reversed) gives the desired limiting behavior.

To treat the two estimates simultaneously, we suppose for the time being that $h, h' \in \{0, \dots, m-1\}$, but we do not assume $h \neq h'$. From Theorem 5.10 together with (5.7) and (5.8) we have

$$\bar{\nu}^h = \lim_{k \rightarrow \infty} \frac{\gamma_k^h}{c_k^h} = \lim_{k \rightarrow \infty} \frac{\gamma_{km+h}}{A(0, km+h)}.$$

Combining this with (5.1), (5.8), and the estimate in Theorem 5.1, we see that for any i we may take k sufficiently large so that

$$(6.2) \quad \begin{aligned} i(\gamma_0^h, \gamma_{i+1}^h) i(\gamma_i^h, \bar{\nu}^{h'}) &\stackrel{*}{\asymp} i(\gamma_h, \gamma_{(i+1)m+h}) i(\gamma_{im+h}, \frac{\gamma_{km+h'}}{A(0, km+h')}) \\ &\stackrel{*}{\asymp} \frac{A(h, (i+1)m+h) A(im+h, km+h')}{A(0, km+h')} \end{aligned}$$

We will simplify the expression on the right, but the precise formula depends on whether $h' \geq h$ or $h' < h$. From the definition (5.1), the right hand side of (6.2) can be written as

$$\frac{\prod_{r=1}^i b e_{rm+h} \prod_{r=j_0}^{k-1} b e_{rm+h'}}{\prod_{r=1}^{k-1} b e_{rm+h'}} = \frac{\prod_{r=1}^i b e_{rm+h}}{\prod_{r=1}^{j_0-1} b e_{rm+h'}}$$

where $j_0 = i + 1$ if $h' \geq h$ and $j_0 = i + 2$ if $h' < h$. Therefore, from (6.2) we can write

$$(6.3) \quad i(\gamma_0^h, \gamma_{i+1}^h) i(\gamma_i^h, \bar{\nu}^{h'}) \stackrel{*}{\asymp} \begin{cases} \prod_{r=1}^i \frac{e_{rm+h}}{e_{rm+h'}} & h' \geq h \\ \frac{1}{b e_{m+h'}} \prod_{r=1}^i \frac{e_{rm+h}}{e_{(r+1)m+h'}} & h' < h \end{cases}$$

Now observe that when $h' = h$, this becomes

$$i(\gamma_0^h, \gamma_{i+1}^h) i(\gamma_i^h, \bar{\nu}^h) \stackrel{*}{\asymp} 1,$$

proving the first of the two required equations. So, suppose $h \neq h'$. Then each of the i terms in the product is bounded above by a^{-1} since the index for the denominator is greater than that of the numerator, and $e_l \geq a e_{l-1}$ for all $l \geq 1$. Thus we have

$$i(\gamma_0^h, \gamma_{i+1}^h) i(\gamma_i^h, \bar{\nu}^{h'}) \stackrel{*}{\prec} a^{-i}$$

where when $h' < h$, we have absorbed the constant $b e_{m+h'}$ into the multiplicative error since $m + h' < 2m$. Letting i tend to infinity, we arrive at the second of our required estimates, and have thus completed the proof. \square

We immediately obtain the following

Corollary 6.2. *The lamination ν is nonuniquely ergodic.*

In fact, Theorem 6.1 implies the main desired result of this section in a special case.

Corollary 6.3. *If $m = \xi(S)$, then $\bar{\nu}^0, \dots, \bar{\nu}^{m-1}$ are ergodic and span the simplex of measures on ν .*

Proof. The theorem guarantees that for each $h = 0, \dots, m - 1$, there is a uniquely ergodic measure with positive coefficient for $\bar{\nu}^h$, and zero coefficient for all other $\bar{\nu}^{h'}$. We can reorder the uniquely ergodic measures so that this is $\bar{\mu}^h$. Since $\xi(S) - 1$ is the maximum dimension of any simplex of measures on a nonuniquely ergodic lamination on the surface S , and since $m = \xi(S)$, it follows that $\bar{\nu}^h$ is a multiple of $\bar{\mu}^h$, for each $h = 0, \dots, m - 1$. \square

6.1. The general case. In [LM10] Lenzhen and Masur prove that for any nonuniquely ergodic lamination ν the ergodic measures are limits of components of multicurves. More precisely, if $\bar{\mu}^0, \dots, \bar{\mu}^{d-1}$ are the ergodic measures, they show that there exists a sequence of multi-curves $\{\sigma_k\}_{k=1}^\infty$ and a partition $\sigma_k = \sigma_k^0 \cup \dots \cup \sigma_k^{d-1}$ so that for each $j = 0, \dots, d-1$

$$\lim_{k \rightarrow \infty} [\sigma_k^j] = [\bar{\mu}^j]$$

in $\mathcal{PML}(S)$. In this section we will recap the relevant features of their construction and apply it to our situation.

We continue to assume $\{\gamma_l\}_{l=0}^\infty$ satisfies \mathcal{P} and that $\{\gamma_k^h\}_{k=0}^\infty$ for $h = 0, \dots, m-1$ is our partition into m sequences, so that

$$(6.4) \quad \sigma_k = \gamma_k^0 \cup \dots \cup \gamma_k^{m-1}$$

is a multi-curve. Our goal is to show that if $\{\sigma_k\}_{k=0}^\infty$ is defined by (6.4), then this serves as a ‘‘Lenzhen-Masur’’ sequence for ν . That is, the ergodic measure are (projective) limits of the m sequences $\{\gamma_k^h\}_{k=0}^\infty$, for $h = 0, \dots, m-1$. Consequently, we will conclude that the dimension of the cone of measures on ν is m , and $\bar{\nu}^0, \dots, \bar{\nu}^{m-1}$ are the ergodic components by Theorem 5.10 and Theorem 6.1.

Let $\bar{\mu}^0, \dots, \bar{\mu}^{d-1}$ be the ergodic measures on ν and set

$$\bar{\mu} = \sum_{j=0}^{d-1} \bar{\mu}^j \quad \text{and} \quad \bar{\gamma} = \sum_{j=0}^{m-1} \gamma_j = \sum_{h=0}^{m-1} \gamma_0^h.$$

Here we are viewing the curves in the sum on the right as measured laminations with transverse counting measure on each curve. We choose a normalization for the measures $\bar{\mu}^j$ so that $i(\bar{\gamma}, \bar{\mu}) = 1$. According to [GM91], there is a unique complex structure on S from a marked Riemann surface $S \rightarrow X$ and unit area holomorphic quadratic differential q on X with at most simple poles at the punctures, so that the vertical foliation $|dx|$ is $\bar{\mu}$ and the horizontal foliation $|dy|$ is $\bar{\gamma}$. Area in the q -metric is computed by integrating $d\bar{\mu}|dy|$. We will also be interested in the measure obtained by integrating $d\bar{\mu}_j|dy|$ for each $j = 0, \dots, d-1$, which we denote by Area_j .

Next let g denote the Teichmüller geodesic defined by q . We will write $g(t) = [f_t: X \rightarrow X(t)]$ where $X(t)$ is the terminal Riemann surface, or $g(t) = [f_t: (X, q) \rightarrow (X(t), q(t))]$, where $q(t)$ is the terminal quadratic differential. Note that since ν is a nonuniquely ergodic lamination by Masur’s criterion [Mas92] the geodesic g is divergent in the moduli space. The vertical and horizontal measure of a curve γ is denoted $v_{q(t)}(\gamma)$ and $h_{q(t)}(\gamma)$, which are precisely the intersection numbers with the horizontal and vertical foliations of $q(t)$, respectively. These are given by

$$v_{q(t)}(\gamma) = e^{-t}i(\gamma, |dy|) = e^{-t}i(\gamma, \bar{\gamma}) \quad \text{and} \quad h_{q(t)}(\gamma) = e^t i(\gamma, |dx|) = e^t i(\gamma, \bar{\mu}).$$

From this it follows that the natural area measure from $q(t)$ is the push forward of the area measure from q . Likewise, this area naturally decomposes

as the push forward of the measures Area_j , for $j = 0, \dots, d-1$. Consequently, we will often confuse a subset of X and its image in $X(t)$ and will simply write Area and Area_j in either X or $X(t)$.

Given $\epsilon > \epsilon' > 0$, an (ϵ', ϵ) -thick subsurface of $(X(t), q(t))$ is a compact surface Y and a continuous map $Y \rightarrow X(t)$, injective on the interior of Y with the following properties.

- (1) The boundary of Y is sent to a union of $q(t)$ -geodesics, each with extremal length less than ϵ' in $X(t)$.
- (2) If Y is not an annulus, then every non-peripheral curve in Y has $q(t)$ -length at least ϵ and Y has no peripheral Euclidean cylinders.
- (3) If Y is an annulus, then it is a maximal Euclidean cylinder.

As an abuse of notation, we will write $Y \subset X$, although Y is only embedded on its interior. An (ϵ', ϵ) -decomposition of $(X(t), q(t))$ is a union of (ϵ', ϵ) -thick subsurfaces $Y_1, \dots, Y_r \subset X(t)$ with pairwise disjoint interiors. We note that $X(t)$ need not be the union of these subsurfaces.

The key results from [LM10] we will need are summarized in the following theorem.

Theorem 6.4 (Lenzhen-Masur). *With the assumptions on the Teichmüller geodesic g above, there exists $\epsilon > 0$ and $B > 0$ with the following property. Given any sequence of times $t_k \rightarrow \infty$, there exists a subsequence (still denoted $\{t_k\}$), a sequence of subsurfaces $Y_0(t_k), \dots, Y_{d-1}(t_k)$ in $X(t_k)$, and a sequence $\epsilon_k \rightarrow 0$, so that for all $k \geq 1$*

- (1) $Y_0(t_k), \dots, Y_{d-1}(t_k)$ is an (ϵ_k, ϵ) -thick decomposition,
- (2) $\text{Area}_j(Y_j^0(t_k)) > B$ for all $0 \leq j \leq d-1$ and for any component $Y_j^0(t_k) \subset Y_j(t_k)$,
- (3) $\text{Area}_j(Y_i(t_k)) < \epsilon_k$ for all $0 \leq i, j \leq d-1$ with $i \neq j$, and
- (4) $\text{Area}(X(t_k) - (Y_0(t_k) \cup \dots \cup Y_{d-1}(t_k))) < \epsilon_k$.

The bulk of this theorem comes from Proposition 1 of [LM10]. More precisely, in the proof of Proposition 1 given in [LM10], the authors produce a sequence of subsurface $\{Y(t_k)\}$ whose components give an (ϵ_k, ϵ) -thick decomposition so that each component has area uniformly bounded away from zero, so that the areas of the complements tend to zero. For each ergodic measure $\bar{\mu}^j$ the authors then find subsurfaces $Y_i(t_k)$ so that $\text{Area}_j(Y_i(t_k)) \rightarrow 0$ as $k \rightarrow \infty$ if $i \neq j$ (see inequality (16) from [LM10] and its proof). This proves (1), (3), and (4). Since $\text{Area} = \sum_j \text{Area}_j$, (2) follows as well.

From the thick decomposition, Lenzhen and Masur construct the required sequence of multi-curves $\{\sigma_k\}$ in the proof of their Theorem C in [LM10]. Each σ_k decomposes as $\sigma_k = \sigma_k^0 \cup \dots \cup \sigma_k^{d-1}$ where σ_k^j contains an appropriately chosen component in each component of $Y_j(t_k)$. The key feature for us is that in each annular component, the authors choose the core curve (see the final paragraph of the proof of Theorem C in [LM10]).

To apply this construction, we will need the following two lemmas. First, for a curve γ and $t \geq 0$, let $\text{cyl}_t(\gamma) \subset X(t)$ denote the (possibly degenerate) maximal Euclidean cylinder foliated by $q(t)$ -geodesic representatives of γ . We note that $\text{cyl}_t(\gamma) = f_t(\text{cyl}_0(\gamma))$.

Lemma 6.5. *With the assumptions on the Teichmüller geodesic g above, there exists $B_0 > 0$ so that $\text{Area}(\text{cyl}_0(\gamma_k^h)) > B_0$ for all $h = 0, \dots, m-1$ and k sufficiently large. Furthermore, for each h , there exists $\{t_k\}_{k=0}^\infty$ so that $\text{Ext}_{X(t_k)}(\gamma_k^h) \rightarrow 0$ and $t_k \rightarrow \infty$ as $k \rightarrow \infty$.*

Lemma 6.6. *Given any sequence $t_k \rightarrow \infty$, suppose $Y(t_k) \subset X(t_k)$ is a sequence of components of the sequence of (ϵ_k, ϵ) -thick decompositions of $X(t_k)$ as in Theorem 6.4. Then after passing to a subsequence, still indexed by k , there exists $0 \leq h \leq m-1$ so that $Y(t_k)$ contains (or is equal to) $\text{cyl}_{t_k}(\gamma_{n_k}^h)$ for some subsequence $\{\gamma_{n_k}^h\}$ of $\{\gamma_l^h\}$.*

We postpone the proofs of these two lemmas temporarily and use them to prove the main result of this section.

Theorem 6.7. *Suppose that $\{\gamma_l\}_{l=0}^\infty$ satisfies \mathcal{P} and that $\{\gamma_k^h\}_{k=0}^\infty$, $h = 0, \dots, m-1$, is the partition into m subsequences with $\lim_{k \rightarrow \infty} \gamma_k^h = \bar{\nu}^h$, all supported on ν . Then $\bar{\nu}^0, \dots, \bar{\nu}^{m-1}$ are the distinct vertices of the simplex of measures on ν . In particular, each $\bar{\nu}^h$ is ergodic.*

Proof. Let $\bar{\mu}^0, \dots, \bar{\mu}^{d-1}$ be ergodic measures spanning the simplex of measures on ν . Fix $0 \leq h \leq m-1$, and let B_0 and the sequence $\{t_k\}$ be from Lemma 6.5. By Theorem 6.4 there exists a subsequence, also denoted $\{t_k\}$, $\epsilon, B, \epsilon_k > 0$, and a (ϵ_k, ϵ) -thick decomposition $Y_0(t_k), \dots, Y_{d-1}(t_k)$ of $X(t_k)$ satisfying the conditions in that Theorem. Since $\text{Ext}_{X(t_k)}(\gamma_k^h) \rightarrow 0$ it follows that for sufficiently large k , γ_k^h cannot be contained as a non-peripheral curve in a non-annular component of any $Y_j(t_k)$, nor can it have nonzero intersection number with $\partial Y_j(t_k)$, for any j . So $\text{cyl}_{t_k}(\gamma_k^h)$ is either equal to an annular component of some $Y_j(t_k)$ or else it is in the complement $X(t_k) - (Y_0(t_k) \cup \dots \cup Y_{d-1}(t_k))$. Since the area of the complement tends to 0 with k , but the cylinders have area bounded below by B_0 , after passing to a subsequence, there exists j so that $\text{cyl}_{t_k}(\gamma_k^h)$ is a component of $Y_j(t_k)$.

As explained above, in the construction of $\sigma_k = \sigma_k^0 \cup \dots \cup \sigma_k^{d-1}$ from [LM10], we will have $\gamma_k^h \subset \sigma_k^j$ for all k sufficiently large. After passing to yet another subsequence, we can assume that the components of σ_k^j all projectively converge to projective classes of measures that sum to $\bar{\mu}^j$. Since $\bar{\mu}^j$ is ergodic, each of the limits, including the limit $\bar{\nu}^h$ of $\{\gamma_k^h\}$, is equal to (a multiple of) $\bar{\mu}^j$. Therefore, $\bar{\nu}^h$ is ergodic. Since h was arbitrary, it follows that $\bar{\nu}^0, \dots, \bar{\nu}^{m-1}$ are all ergodic measures.

Next, fix $0 \leq j \leq d-1$, and let $Y(t_k) \subset Y_j(t_k)$ be a component for each k . According to Lemma 6.6, we can pass to a further subsequence and find $0 \leq h' \leq m-1$ so that $Y(t_k)$ contains $\text{cyl}_{t_k}(\gamma_{n_k}^{h'})$ for all k . Since

$\text{Area}(\text{cyl}_0(\gamma_{n_k}^{h'})) = \text{Area}(\text{cyl}_{t_k}(\gamma_{n_k}^{h'})) > B_0$, there exists $0 \leq j' \leq d-1$ so that

$$(6.5) \quad \text{Area}_{j'}(\text{cyl}_{t_k}(\gamma_{n_k}^{h'})) > B_0/d.$$

From Theorem 6.4 parts (2) and (3), we must have $j' = j$.

Now according to Lemma 6.5, we may choose a (possibly different) sequence of times $\{t'_k\}$ so that, as above, $\text{cyl}_{t'_k}(\gamma_{n_k}^{h'})$ is a component of some (ϵ'_k, ϵ') -thick surface $Y_{j''}(t'_k)$ of the (ϵ'_k, ϵ') -thick decomposition obtained from Theorem 6.4 applied to this sequence. By (6.5), we have

$$\text{Area}_{j'}(\text{cyl}_{t'_k}(\gamma_{n_k}^{h'})) = \text{Area}_{j''}(\text{cyl}_{t'_k}(\gamma_{n_k}^{h'})) > B_0/d$$

and so by part (3) of Theorem 6.4 (since $\epsilon'_k \rightarrow 0$), it follows that $j'' = j'$ and since $j' = j$, we have $j'' = j$. Now arguing as above, we see that $\bar{\nu}^{h'}$ and $\bar{\mu}^j$ differ by a scalar multiple.

Combining these two conclusions, we see that every $\bar{\nu}^0, \dots, \bar{\nu}^{m-1}$ is an ergodic measure and that any ergodic measure is a multiple of one of these. Finally, according to Theorem 6.1 these measures are all distinct, and hence $d = m$ and $\bar{\nu}^0, \dots, \bar{\nu}^{m-1}$ are the m ergodic measures spanning the simplex of all measures on ν . \square

6.2. Areas and extremal lengths. The proofs of the two lemmas basically follow from the results of [Raf05], together with the estimates on intersection numbers described at the beginning of this section and subsurface coefficient bounds in §4.1. Let $g(t) = [f_t: (X, q) \rightarrow (X(t), q(t))]$ be the Teichmüller geodesic described above with vertical foliation $\bar{\mu} = \sum \bar{\mu}_i$, the sum of the ergodic measures on ν , and horizontal foliation $|dy| = \bar{\gamma}$.

Suppose $Y \rightarrow X(t)$ is a map of a connected surface into $X(t)$ which is an embedding on the interior, sends the boundary to $q(t)$ -geodesics, and has no peripheral Euclidean cylinders unless Y is itself a Euclidean cylinder (in which case we assume it is maximal). As in the case of thick subsurfaces, we write $Y \subset X(t)$, though we are not assuming that Y is thick. Suppose $Y \subset X(t)$ is a subsurface so that the leaves of the vertical and horizontal foliations intersect Y in arcs. This is the case for $Y = \text{cyl}_t(\gamma_k)$ for all k sufficiently large, as well as any Y for which $\text{Ext}_{X(t)}(\partial Y)$ is small when t is large, and these will be the main cases of interest for us.

As in [Raf05], the surface Y decomposes into a union of *horizontal strips* $Y = H_1(Y) \cup \dots \cup H_r(Y)$ and *vertical strips* $Y = V_1(Y) \cup \dots \cup V_{r'}(Y)$. Each horizontal strip $H_i(Y)$ is the image of map $f_i^H: [0, 1] \times [0, 1] \rightarrow Y$ which is injective on the interior, sends $[0, 1] \times \{s\}$ to an arc of a horizontal leaf with endpoints on ∂Y . Furthermore, the images of the interiors of f_1^H, \dots, f_r^H are required to be pairwise disjoint. Let $\ell_i^H = f_i^H([0, 1] \times \{\frac{1}{2}\})$ be a “core arc” of the strip. Vertical strips are defined similarly (and satisfy the analogous properties for the vertical foliation) as are the core arcs $\ell_1^V, \dots, \ell_{r'}^V$.

Remark 6.8. This is a slight variation on the strip decompositions in [Raf05].

The *width* of a horizontal strip $H_i(Y)$, denoted $w(H_i(Y))$ is the vertical variation of any (or equivalently, every) arc $H_i(\{s\} \times [0, 1])$. The width of a vertical strip, $w(V_i(Y))$, is similarly defined in terms of the horizontal variation. An elementary, but important property of these strips is the following.

Proposition 6.9. *Let $Y \subset X(t)$ be as above. If*

$$Y = H_1(Y) \cup \dots \cup H_r(Y) = V_1(Y) \cup \dots \cup V_{r'}(Y)$$

is a decomposition into maximal horizontal and vertical strips, then

$$v_{q(t)}(\partial Y) = 2 \sum_{i=1}^r w(H_i(Y)) \quad \text{and} \quad h_{q(t)}(\partial Y) = 2 \sum_{i=1}^{r'} w(V_i(Y)).$$

The area of Y can be estimated from this by the inequalities

$$(6.6) \quad \sum_{ij} w(H_i(Y))w(V_j(Y))(i(\ell_i^H, \ell_j^H) - 2) \leq \text{Area}(Y) \\ \leq \sum_{i,j} w(H_i(Y))w(V_j(Y))(i(\ell_i^H, \ell_j^H) + 2)$$

To see this, we note that the area of Y is the sum of the areas of the horizontal (or vertical) strips. Every time $V_j(Y)$ crosses $H_i(Y)$, it does so in a rectangle, which contains a unique point of intersection $\ell_i^H \cap \ell_j^V$, **except**, near the ends of $H_i(Y)$ where we might not see an entire rectangle (and consequently we may or may not see a point of $\ell_i^H \cap \ell_j^V$). We may also have an intersection point in $\ell_i^H \cap \ell_j^V$ that does not come in a complete rectangle (but only part of a rectangle). Adding and subtracting 2 to the intersection number accounts for the ends of $H_i(Y)$, and summing gives the bounds.

If Y is non-annular, then note that

$$\sum i(\ell_i^H, \ell_j^V) + 2 \prec i(\pi_Y(\bar{\gamma}), \pi_Y(\nu)).$$

To see this, we note that the horizontal foliation (for example) is $\bar{\gamma}$ and $\pi_Y(\bar{\gamma})$ is basically obtained from the arcs ℓ_i^H by surgering with arcs from the boundary (see also Lemma 3.8 from [Raf05]). Combining this inequality with the upper bound in (6.6) and Proposition 6.9 we obtain

$$(6.7) \quad \text{Area}(Y) \prec h_{q(t)}(\partial Y)v_{q(t)}(\partial Y)i(\pi_Y(\bar{\gamma}), \pi_Y(\nu)).$$

Now suppose that $Y = \text{cyl}_t(\gamma)$ is a maximal Euclidean cylinder with core curve γ . Then there is a decomposition into strips with just one horizontal strip $H(Y)$ and one vertical strip $V(Y)$ and core arcs ℓ^H and ℓ^V , respectively. In this case, the intersection number $i(\ell^H, \ell^V)$ is just $d_Y(\bar{\gamma}, \nu)$ up to an additive constant (of at most 4—again, see Lemma 3.8 of [Raf05]). Therefore, the bounds in (6.6) together with Proposition 6.9 implies

$$(6.8) \quad \frac{4 \text{Area}(\text{cyl}_0(\gamma))}{h_{q(t)}(\gamma)v_{q(t)}(\gamma)} = \frac{4 \text{Area}(\text{cyl}_0(\gamma))}{i(\gamma, \bar{\gamma})i(\gamma, \bar{\mu})} \stackrel{\pm}{\asymp} d_\gamma(\bar{\gamma}, \nu).$$

In particular, if $d_\gamma(\bar{\gamma}, \nu)$ is large, then

$$\text{Area}(\text{cyl}_0(\gamma_k^h)) \stackrel{*}{\asymp} h_{q(t)}(\gamma)v_{q(t)}(\gamma)d_\gamma(\bar{\gamma}, \nu) = i(\gamma, \bar{\gamma})i(\gamma, \bar{\mu})\gamma(\bar{\gamma}, \nu).$$

The balance time of γ along the Teichmüller geodesic g is the unique $t \in \mathbb{R}$ so that

$$v_{q(t)}(\gamma) = h_{q(t)}(\gamma).$$

Consider $Y = \text{cyl}_{t(\gamma)}(\gamma)$ at the balance time of γ , together with the horizontal and vertical strips $H(Y)$ and $V(Y)$, respectively. In this situation, the rectangles of intersections between $H(Y)$ and $V(Y)$ are actually squares. We can estimate the modulus of Y , which is the ratio of the length to the circumference using these squares. Specifically, we note that the circumference of Y is precisely the length of the diagonal of a square, while the length of Y is approximately half the number of squares, times the length of a diagonal. Since the number of squares is $|\ell^H \cap \ell^V| \stackrel{\pm}{\asymp} d_\gamma(\bar{\gamma}, \nu)$, we see that the modulus is $2d_\gamma(\bar{\gamma}, \nu)$, up to a uniform additive error. When $d_\gamma(\bar{\gamma}, \nu)$ is sufficiently large, the reciprocal of this modulus provides an upper bound for the extremal length

$$(6.9) \quad \text{Ext}_{t(\gamma)}(\gamma) \stackrel{*}{\prec} \frac{1}{d_\gamma(\bar{\gamma}, \nu)}.$$

We note that this estimate was under the assumption that $\text{cyl}_0(\gamma)$ was a non-degenerate annulus. In fact, if $d_\gamma(\bar{\gamma}, \nu)$ is sufficiently large (e.g. at least 5), then $\text{cyl}_0(\gamma)$ is indeed nondegenerate.

Proof of Lemma 6.5. First note that for k sufficiently large, Proposition 4.5 tells us that

$$d_{\gamma_k}(\bar{\gamma}, \nu) = d_{\gamma_k}(\gamma_0 \cup \dots \cup \gamma_{m-1}, \nu) \asymp e_k.$$

Let $t(\gamma_k)$ be the balance time for γ_k . Since $e_k \rightarrow \infty$ as $k \rightarrow \infty$, (6.9) implies $\text{Ext}_{X(t(\gamma_k))}(\gamma_k) \rightarrow 0$ as $k \rightarrow \infty$. To complete the proof of the last statement of the lemma, we need only observe that as $k \rightarrow \infty$, $t(\gamma_k) \rightarrow \infty$. This follows from the fact that every accumulation point of $\{\gamma_k\}_{k=0}^\infty$ in $\mathcal{PML}(S)$ is supported on ν .

Now fix $0 \leq h \leq m-1$, and we use (6.8) to estimate $\text{Area}(\text{cyl}_0(\gamma_k^h))$, bounding it away from zero uniformly. For this, we first note that by a calculation similar to the one used to deduce (6.3) we have

$$i(\gamma_0^h, \gamma_k^h)i(\gamma_k^h, \bar{\nu}^h) \stackrel{*}{\asymp} \frac{1}{e_{km+h}}$$

Since $\bar{\gamma} = \gamma_0^0 \cup \dots \cup \gamma_0^{m-1}$ and since $\bar{\nu}^h$ has positive coefficients on some of the ergodic measures $\bar{\mu}^0, \dots, \bar{\mu}^{d-1}$, while $\bar{\mu}$ has positive coefficients on all of these measures, we have

$$i(\bar{\gamma}, \gamma_k^h)i(\gamma_k^h, \bar{\mu})d_{\gamma_k^h}(\bar{\gamma}, \nu) \stackrel{*}{\succ} \frac{1}{e_{km+h}}e_{km+h} = 1.$$

Combining this with (6.8) we have

$$\text{Area}(\text{cyl}_0(\gamma_k^h)) \succ^* \frac{1}{e_{km+h}} e_{km+h} = 1.$$

□

We now turn to the

Proof of Lemma 6.6. Suppose that $t_k \rightarrow \infty$ is a sequence of times, $Y(t_k) \subset X(t_k)$ is a sequence of subsurfaces with $q(t)$ -geodesic boundary, embedded on the interior and having no peripheral Euclidean cylinders, unless Y is itself a Euclidean cylinder in which case we assume it is a maximal Euclidean cylinder. We further assume that $\text{Ext}_{X(t_k)}(\partial Y(t_k)) \rightarrow 0$. We pass to a subsequence, also denoted $\{t_k\}$, and assume that $Y(t_k)$ that does not contain any cylinder $\text{cyl}_{t_k}(\gamma_{n_k})$, for any subsequence $\{\gamma_{n_k}\}$. To prove the lemma, it suffices to prove that $\text{Area}(Y(t_k)) \rightarrow 0$, for this implies that such subsurfaces $Y(t_k)$ cannot be a component of any $Y_j(t_k)$ from Theorem 6.4.

Decompose the sequence into an annular subsequence and non-annular subsequence, and we consider each case separately. For the non-annular subsurfaces, we bound the area of $Y(t_k)$ using the inequality (6.7). Specifically, we note that since no γ_k is homotopic to a nonperipheral curve in $Y(t_k)$ (otherwise $\text{cyl}_{t_k}(\gamma_k)$ would be contained in $Y(t_k)$), Proposition 4.5 provides a uniform bound for $d_W(\bar{\gamma}, \nu)$ for all subsurfaces $W \subset Y(t_k)$. By Theorem 2.7, follows that $i(\pi_Y(\bar{\gamma}), \pi_Y(\nu))$ is uniformly bounded. Since the extremal length of $\partial Y(t_k)$ is tending to zero, so is the $q(t_k)$ -length, and hence the horizontal and vertical variations:

$$v_{q(t_k)}(\gamma_k), h_{q(t_k)}(\gamma_k) \rightarrow 0$$

as $k \rightarrow \infty$. Combining this with (6.7) proves $\text{Area}(Y(t_k)) \rightarrow 0$, as required.

The annular case is similar: Again by Proposition 4.5 since the core curve α_k of $Y(t_k)$ is not any curve from the sequence $\{\gamma_l\}$, we have that $d_{\alpha_k}(\bar{\gamma}, \bar{\nu})$ is uniformly bounded, while the horizontal and vertical variations of α_k tend to zero. Appealing to (6.8) proves that $\text{Area}(Y(t_k)) \rightarrow 0$ as $k \rightarrow \infty$ in this case, too. □

7. CONSTRUCTIONS

In this section we provide examples of sequences of curves satisfying \mathcal{P} , and hence to which the results of Sections 3-6 apply.

7.1. Basic setup. Consider a surface S and pairwise disjoint, non-isotopic curves $\gamma_0, \dots, \gamma_{m-1}$. For each k , let $\Gamma_k = (\gamma_0 \cup \dots \cup \gamma_{m-1}) - \gamma_k$, and let X_k be the component of S cut along Γ_k containing γ_k . For each k we assume

- (1) ∂X_k contains both γ_{k+1} and γ_{k-1} (with indices taken modulo m),
- (2) we have chosen $f_k: S \rightarrow S$ a *fixed* homeomorphism which is the identity on $S \setminus X_k$, and pseudo-Anosov on X_k ,
- (3) the translation distance of $\mathcal{D}_{\gamma_k}^r f_k$ on the arc-and-curve graph $\mathcal{AC}(X_k)$ is at least 16 for any $r \in \mathbb{Z}$, and

(4) there is some $b > 0$ so that $i(\gamma_k, f_k(\gamma_k)) = b$, independent of k .

For $0 \leq k, h \leq m-1$, let $\mathcal{J}(k, h)$ be the interval from k to h , mod m . This means that if $k < h$ then $\mathcal{J}(k, h) = \{k, k+1, \dots, h\}$ is the interval in \mathbb{Z} from k to h , while if $h < k$, then

$$\mathcal{J}(k, h) = \{k, k+1, \dots, m-1, 0, \dots, h\}.$$

If $k = h$, then $\mathcal{J}(k, h) = \{k\} = \{h\}$.

For any $0 \leq k, h \leq m-1$, set

$$X_{k,h} = \bigcup_{l \in \mathcal{J}(k,h)} X_l.$$

If $k = h$, note that $X_{k,h} = X_k = X_h$. In general, $X_{k,h}$ is the component of S cut along $\Gamma_{k,h} = \gamma_{h+1} \cup \dots \cup \gamma_{k-1}$ containing all the curves $\gamma_k, \dots, \gamma_h$. That there is such a component follows inductively from the fact that $\gamma_{l \pm 1} \subseteq \partial X_l$, with indices taken mod m .

We also define

$$F_{k,h} = f_k \circ f_{k+1} \circ \dots \circ f_h.$$

where we are composing f_l over $l \in \mathcal{J}(k, h)$. Because f_l is supported on X_l , it follows that for all $0 \leq k, h \leq m-1$,

$$\gamma_k, \dots, \gamma_h, F_{k,h}(\gamma_h) \subset X_{k,h}.$$

In fact, the first and last curves in this sequence fill $X_{k,h}$.

Lemma 7.1. *For each $0 \leq k, h \leq m-1$*

$$\{\gamma_k, F_{k,h}(\gamma_h)\}$$

fills $X_{k,h}$. In particular, $i(\gamma_l, F_{k,h}(\gamma_h)) \neq 0$ for all $l \in \mathcal{J}(k, h)$.

Remark 7.2. In the case $k = h+1 \pmod{m}$, we note that $X_{h+1,h} = S$ and the lemma states that

$$\{\gamma_{h+1}, F_{h+1,h}(\gamma_h)\} = \{\gamma_k, f_k f_{k+1} \dots f_h(\gamma_h)\}$$

fills S . We also observe that for all $j \in \mathcal{J}(k, h)$, $X_{k,j} \subset X_{k,h}$. It follows that $\gamma_k, \gamma_{k+1}, \dots, \gamma_h$ and $F_{k,k}(\gamma_k), \dots, F_{k,h}(\gamma_h)$ are contained in $X_{k,h}$.

In the following proof, we write $\pi_{k,h}(\delta)$ for the arc-projection to $\mathcal{AC}(X_{k,h})$ of a curve δ . This is just the isotopy class of arcs/curves of δ intersected with $X_{k,h}$. Likewise, $d_{k,h}(\delta, \delta')$ is the distance between $\pi_{k,h}(\delta)$ and $\pi_{k,h}(\delta')$ in $\mathcal{AC}(X_{k,h})$. We similarly define π_k and d_k for the case $k = h$.

Proof. The last statement follows from the first assertion since, for all $l \in \mathcal{J}(k, h)$, $i(\gamma_l, \gamma_k) = 0$, and so assuming $\{\gamma_k, F_{k,h}(\gamma_h)\}$ fills, we must have $i(\gamma_l, F_{k,h}(\gamma_h)) \neq 0$.

The conditions on the curves and homeomorphisms are symmetric under cyclic permutation of the indices, so it suffices to prove the lemma for $h = m-1$ and $0 \leq k \leq h$ (which is slightly simpler notationally). We write $j = h-k$ and must prove that $\{\gamma_{h-j}, F_{h-j,h}(\gamma_h)\}$ fills $X_{h-j,h}$. We prove this by induction on j .

The base case is $j = 0$, in which case we are reduced to proving that $\{\gamma_h, f_h(\gamma_h)\}$ fills X_h . This follows from the fact that f_h has translation distance at least 16 on $\mathcal{AC}(X_h)$, and hence $d_h(\gamma_h, f_h(\gamma_h)) \geq 16$.

Suppose that for some $0 < j \leq h$, $\{\gamma_{h+1-j}, F_{h+1-j,h}(\gamma_h)\}$ fill $X_{h+1-j,h}$, and we must prove that $\{\gamma_{h-j}, F_{h-j,h}(\gamma_h)\}$ fills $X_{h-j,h}$.

Note that since $\gamma_{h-j+1} \subset \partial X_{h-j}$, and $i(\gamma_{h-j+1}, F_{h+1-j,h}(\gamma_h)) \neq 0$ (because they fill $X_{h+1-j,h}$), it follows that $F_{h+1-j,h}(\gamma_h)$ has nontrivial projection to X_{h-j} . On the other hand, because γ_{h-j} is disjoint from $X_{h+1-j,h}$ (it is in fact a boundary component), it follows that $i(\gamma_{h-j}, F_{h+1-j,h}(\gamma_h)) = 0$, hence $d_{h-j}(\gamma_{h-j}, F_{h+1-j,h}(\gamma_h)) = 1$. Since f_{h-j} translates by at least 16 on $\mathcal{AC}(X_{h-j})$, it follows that

$$\begin{aligned} d_{h-j}(F_{h-j,h}(\gamma_h), \gamma_{h-j}) &= d_{h-j}(f_{h-j}(F_{h+1-j,h}(\gamma_h)), \gamma_{h-j}) \\ &\geq d_{h-j}(f_{h-j}(F_{h+1-j,h}(\gamma_h)), F_{h+1-j,h}(\gamma_h)) \\ &\quad - d_{h-j}(F_{h+1-j,h}(\gamma_h), \gamma_{h-j}) \\ &\geq 16 - 1 = 15. \end{aligned}$$

Now suppose $\{\gamma_{h-j}, F_{h-j,h}(\gamma_h)\}$ does not fill $X_{h-j,h}$. Let δ be an essential curve in $X_{h-j,h}$ which is disjoint from both γ_{h-j} and $F_{h-j,h}(\gamma_h)$. Observe that δ cannot intersect the subsurface X_{h-j} essentially, for otherwise

$$d_{h-j}(\gamma_{h-j}, F_{h-j,h}(\gamma_h)) \leq d_{h-j}(\gamma_{h-j}, \delta) + d_{h-j}(\delta, F_{h-j,h}(\gamma_h)) \leq 2$$

a contradiction.

Therefore, δ is contained in $X_{h-j,h} - X_{h-j} \subset X_{h+1-j,h}$. We first claim that δ must be an essential curve in $X_{h+1-j,h}$. If not, then it is contained in the boundary. However, any boundary component of $X_{h+1-j,h}$ which is essential in $X_{h-j,h}$ is contained (and essential) in X_{h-j} . This is a contradiction.

Now since δ is essential in $X_{h+1-j,h}$, by the hypothesis of the induction we have

$$0 \neq i(\delta, \gamma_{h+1-j}) + i(\delta, F_{h+1-j,h}(\gamma_h)) = i(\delta, \gamma_{h+1-j}) + i(\delta, F_{h-j,h}(\gamma_h)).$$

The last equality follows from the fact that $F_{h-j,h}$ differs from $F_{h+1-j,h}$ only in X_{h-j} , which is disjoint from δ . Finally, we note that $\gamma_{h+1-j} \subseteq \partial X_{h-j}$, and hence $i(\delta, \gamma_{h+1-j}) = 0$. Consequently,

$$i(\delta, F_{h-j,h}(\gamma_h)) \neq 0$$

contradicting our choice of δ . Therefore, $\{\gamma_{h-j}, F_{h-j,h}(\gamma_h)\}$ fills $X_{h-j,h}$. This completes the induction, and hence the proof of the lemma. \square

Lemma 7.3. *For all $0 \leq k \leq m-1$,*

$$i(\gamma_k, F_{k,k-1}f_k(\gamma_k)) = i(\gamma_k, f_k f_{k+1} \cdots f_{k-1} f_k(\gamma_k)) \neq 0.$$

Proof. We recall from the previous proof that $\{\gamma_{k+1}, F_{k+1,k}(\gamma_k)\}$ not only fills S , but satisfies

$$d_{k+1}(\gamma_{k+1}, F_{k+1,k}(\gamma_k)) \geq 15.$$

Since $\gamma_{k+1} \subseteq \partial X_k$ and $\gamma_k \subseteq \partial X_{k+1}$ and X_k and X_{k+1} overlap, Theorem 2.9 (see also Remark 2.10) implies

$$d_k(\gamma_k, F_{k+1,k}(\gamma_k)) \leq 4.$$

Since f_k translates at least 16 on $\mathcal{AC}(X_k)$, it follows that

$$\begin{aligned} d_k(\gamma_k, f_k F_{k+1,k}(\gamma_k)) &\geq d_k(F_{k+1,k}(\gamma_k), f_k F_{k+1,k}(\gamma_k)) - d_k(\gamma_k, F_{k+1,k}(\gamma_k)) \\ &\geq 16 - 4 \geq 12. \end{aligned}$$

Since $f_k F_{k+1,k} = F_{k,k-1} f_k$, the lemma follows. \square

7.2. General construction. Let $\{e_k\}_{k=0}^\infty$ be a sequence of integers satisfying Inequality (3.1) for $a > 2$ sufficiently large as so as to satisfy (5.6) and hence (5.3) in Theorem 5.1 (see the convention at the end of Section 5.1).

For $k \geq 0$, let $\bar{k} \in \{0, \dots, m-1\}$ be the residue mod m , and for $k \geq m$ define

$$\mathcal{D}_k = \mathcal{D}_{\gamma_{\bar{k}}}^{e_{k-m}} \text{ and } \phi_k = \mathcal{D}_k f_{\bar{k}}.$$

The sequence of curves $\{\gamma_k\}_{k=0}^\infty$ is defined as follows.

- (1) The first m curves are $\gamma_0, \dots, \gamma_{m-1}$, as above.
- (2) For $k \geq m$, set

$$\gamma_k = \phi_m \phi_{m+1} \cdots \phi_k(\gamma_{\bar{k}}).$$

Remark 7.4. We could have avoided having the first m curves as special cases and alternatively defined a sequence $\{\delta_k\}_{k \geq 0}$ by $\delta_k = \phi_0 \cdots \phi_k(\gamma_{\bar{k}})$ for all $k \geq 0$. This sequence differs from ours by applying the homeomorphism $\phi_0 \cdots \phi_{m-1}$. This is a useful observation when it comes to describing consecutive elements in the sequence, but our choice allows us to keep $\gamma_0, \dots, \gamma_{m-1}$ as the first m curves.

Proposition 7.5. *With the conditions above, the sequence $\{\gamma_k\}_{k=0}^\infty$ satisfies \mathcal{P} for some $0 < b_1 \leq b \leq b_2$ (where b is the constant assumed from the start).*

To simplify the proof, we begin with

Lemma 7.6. *For any $2m$ consecutive curves $\gamma_{k-m}, \dots, \gamma_{k+m-1}$, there is a homeomorphism $H_k : S \rightarrow S$ taking these curves to the curves*

$$\gamma_{\bar{k}}, \dots, \gamma_{\overline{k+m-1}}, f_{\bar{k}}(\gamma_{\bar{k}}), \dots, f_{\bar{k}} \cdots f_{\overline{k+m-1}}(\gamma_{\overline{k+m-1}})$$

(in the same order). Furthermore, the homeomorphism can be chosen to take γ_{k+m} to

$$\mathcal{D}_{k_{\bar{k}}(\gamma_{\bar{k}})}^{e_k}(f_{\bar{k}} \cdots f_{\overline{k+m-1}} f_{\bar{k}}(\gamma_{\bar{k}})).$$

Proof. We prove the lemma assuming $k \geq 2m$ to avoid special cases (the general case can be easily derived from Remark 7.4, for example). We define

$$H_k = (\phi_m \cdots \phi_{k-1} \mathcal{D}_k \mathcal{D}_{k+1} \cdots \mathcal{D}_{k+m-1})^{-1}.$$

Let $h, h' \in \{0, \dots, m-1\}$ and note that since $i(\gamma_h, \gamma_{h'}) = 0$, $\mathcal{D}_{\gamma_{h'}}(\gamma_h) = \gamma_h$. Furthermore, if $h \neq h'$, from the fact that f_h is supported on X_h and $\gamma_{h'}$ is disjoint from X_h we easily deduce $\mathcal{D}_{\gamma_{h'}}$ and f_h commute, and $\phi_h(\gamma_{h'}) = \gamma_{h'}$.

From these facts we observe that for $k - m \leq j \leq k - 1$, we have

$$\begin{aligned} H_k^{-1}(\gamma_j) &= \phi_m \cdots \phi_{k-1} \mathcal{D}_k \mathcal{D}_{k+1} \cdots \mathcal{D}_{k+m-1}(\gamma_{\bar{j}}) \\ &= \phi_m \cdots \phi_{k-1}(\gamma_{\bar{j}}) \\ &= \phi_m \cdots \phi_j(\gamma_{\bar{j}}) = \gamma_j, \end{aligned}$$

while for $k \leq j \leq k + m - 1$, we have

$$\begin{aligned} H_k^{-1}(f_{\bar{k}} \cdots f_{\bar{j}}(\gamma_{\bar{j}})) &= \phi_m \cdots \phi_{k-1} \mathcal{D}_k \cdots \mathcal{D}_{k+m-1} f_{\bar{k}} \cdots f_{\bar{j}}(\gamma_j) \\ &= \phi_m \cdots \phi_{k-1} \mathcal{D}_k f_{\bar{k}} \cdots \mathcal{D}_j f_{\bar{j}} \mathcal{D}_{j+1} \cdots \mathcal{D}_{k+m-1}(\gamma_{\bar{j}}) \\ &= \phi_m \cdots \phi_j \mathcal{D}_{j+1} \cdots \mathcal{D}_{k+m-1}(\gamma_{\bar{j}}) \\ &= \phi_m \cdots \phi_j(\gamma_{\bar{j}}) = \gamma_j. \end{aligned}$$

This completes the proof of the first statement.

Next, since $\mathcal{D}_{k+m} = \mathcal{D}_{\gamma_{\bar{k}}}^{e_k}$, we have

$$\begin{aligned} (7.1) \quad f_{\bar{k}} \cdots f_{\overline{k+m-1}} \phi_{k+m}(\gamma_{\bar{k}}) &= f_{\bar{k}} \cdots f_{\overline{k+m-1}} \mathcal{D}_{k+m} f_{\bar{k}}(\gamma_{\bar{k}}) \\ &= f_{\bar{k}} \mathcal{D}_{k+m} f_{\overline{k+1}} \cdots f_{\overline{k+m-1}} f_{\bar{k}}(\gamma_{\bar{k}}) \\ &= f_{\bar{k}} \mathcal{D}_{k+m} f_{\bar{k}}^{-1} f_{\bar{k}} \cdots f_{\overline{k+m-1}} f_{\bar{k}}(\gamma_{\bar{k}}) \\ &= f_{\bar{k}} \mathcal{D}_{\gamma_{\bar{k}}}^{e_k} f_{\bar{k}}^{-1} f_{\bar{k}} \cdots f_{\overline{k+m-1}} f_{\bar{k}}(\gamma_{\bar{k}}) \\ &= \mathcal{D}_{\gamma_{\bar{k}}}^{e_k} f_{\bar{k}} \cdots f_{\overline{k+m-1}} f_{\bar{k}}(\gamma_{\bar{k}}). \end{aligned}$$

Applying H_k^{-1} to the left hand side gives γ_{k+m} , proving the last statement. \square

Proof of Proposition 7.5. For any $2m$ consecutive curves in our sequence, $\gamma_{k-m}, \dots, \gamma_{k+m-1}$, let $H_k: S \rightarrow S$ be the homeomorphism from Lemma 7.6 putting these curves into the standard form described by that lemma. Since H_k sends the first m to $\gamma_{\bar{k}}, \dots, \gamma_{\overline{k+m-1}}$, it follows that these curves are pairwise disjoint. Moreover, the set of all $2m$ curves fills S by Lemma 7.1 and Remark 7.2 (in fact, the first and last alone fill S). Therefore, the sequence satisfies conditions (i) and (ii) of \mathcal{P} .

To prove that condition (iii) is also satisfied we need to define γ'_{k+m} so that $\gamma_{k+m} = \mathcal{D}_{\gamma_{\bar{k}}}^{e_k}(\gamma'_{k+m})$, and verify the intersection conditions. We fix $k \geq 2m$ and define

$$\gamma'_{k+m} = \phi_m \cdots \phi_{m+k-1} f_{\bar{k}}(\gamma_{\bar{k}})$$

(the case of general $k \geq m$ is handled by special cases or by appealing to Remark 7.4). Note that by definition, $\gamma_{k+m} = \phi_m \cdots \phi_{m+k-1} \phi_{m+k}(\gamma_{\bar{k}})$ and applying H_k to γ_k and γ_{k+m} , Lemma 7.6 gives us

$$H_k(\gamma_k) = f_{\bar{k}}(\gamma_{\bar{k}}) \quad \text{and} \quad H_k(\gamma_{k+m}) = \mathcal{D}_{\gamma_{\bar{k}}}^{e_k}(\gamma_{\bar{k}})(f_{\bar{k}} \cdots f_{\overline{k+m-1}} f_{\bar{k}}(\gamma_{\bar{k}}))$$

Then, as in the proof of Lemma 7.6 (compare (7.1)), we have

$$H_k(\gamma'_{k+m}) = f_{\bar{k}} \cdots f_{\overline{k+m-1}} f_{\bar{k}}(\gamma_{\bar{k}})$$

Therefore,

$$H_k(\gamma_{k+m}) = \mathcal{D}_{H_k(\gamma_k)}^{e_k}(H_k(\gamma'_{k+m})) = H_k(\mathcal{D}_{\gamma_k}^{e_k}(\gamma'_{k+m})),$$

so $\gamma_{k+m} = \mathcal{D}_{\gamma_k}^{e_k}(\gamma'_{k+m})$.

To prove the intersection number conditions on $i(\gamma'_{k+m}, \gamma_j)$ from property (iii) of \mathcal{P} , it suffices to prove them for the H_k -images. Thus, for $j \in \{k+1, \dots, k+m-1\}$ we note that by Lemma 7.6, $H_k(\gamma_j) = f_{\bar{k}} \cdots f_{\bar{j}}(\gamma_{\bar{j}})$, and hence

$$\begin{aligned} i(\gamma_j, \gamma'_{k+m}) &= i(f_{\bar{k}} \cdots f_{\bar{j}}(\gamma_{\bar{j}}), f_{\bar{k}} \cdots f_{\overline{k+m-1}} f_{\bar{k}}(\gamma_{\bar{k}})) \\ &= i(\gamma_{\bar{j}}, f_{\overline{j+1}} \cdots f_{\overline{k+m-1}} f_{\bar{k}}(\gamma_{\bar{k}})) \\ &= i(\gamma_{\bar{j}}, \gamma_{\bar{k}}) = 0. \end{aligned}$$

The second-to-last equality is obtained by applying $(f_{\overline{j+1}} \cdots f_{\overline{k+m-1}} f_{\bar{k}})^{-1}$ to both entries, and observing that this fixes $\gamma_{\bar{j}}$ (c.f. the proof of Lemma 7.6).

On the other hand, for $j = k$, the same basic computation shows

$$i(\gamma_k, \gamma'_{k+m}) = i(\gamma_{\bar{k}}, f_{\bar{k}}(\gamma_{\bar{k}})) = b$$

by assumption (4).

Finally, similar calculations show that for $j \in \{k-m, \dots, k-1\}$, by Lemmas 7.1 and 7.3, we have

$$i(\gamma_j, \gamma'_{k+m}) = i(\gamma_{\bar{j}}, f_{\bar{k}} \cdots f_{\overline{k+m-1}} f_{\bar{k}}(\gamma_{\bar{k}})) \neq 0.$$

There are only finitely many possible choices of \bar{j} and \bar{k} , so the values are uniformly bounded between two constants $b_1 < b_2$. Without loss of generality, we may assume $b_1 \leq b \leq b_2$. This completes the proof. \square

While any sequence of curves as above satisfies the conditions in sections in \mathcal{P} from Definition 3.1, we will need one more condition when analyzing the limits of Teichmüller geodesics. It turns out that any construction as above also satisfies this property. We record this property here for later use.

Lemma 7.7. *Suppose the sequence $\{\gamma_k\}_{k=0}^\infty$ is constructed as above. If γ_k, γ_h are any two curves with $m \leq h-k < 2m-1$, then γ_k and γ_h fill a subsurface whose boundary consists entirely of curves in the sequence. Furthermore, for any $k \leq j \leq h$, γ_j is either contained in this subsurface, or is disjoint from it. If $h-k \geq 2m-1$, then γ_k and γ_h fill S .*

Proof. First assume $m \leq h-k \leq 2m-1$. Applying the homeomorphism $H_k: S \rightarrow S$ from Lemma 7.6, γ_k and γ_h are sent to $\gamma_{\bar{k}}$ and $f_{\bar{k}} \cdots f_{\bar{h}}(\gamma_{\bar{h}}) = F_{\bar{k}, \bar{h}}(\gamma_{\bar{h}})$, respectively. This fills the surface $X_{\bar{k}, \bar{h}}$ which has boundary contained in $\gamma_0 \cup \dots \cup \gamma_{m-1}$. By Lemma 7.1 it follows that $H_k^{-1}(X_{\bar{k}, \bar{h}})$ is filled by $\{\gamma_k, \gamma_h\}$ and has boundary in $H_k(\gamma_0) \cup \dots \cup H_k(\gamma_{m-1})$. All the components of this multicurve are in our sequence, as required for the first statement.

For each $k \leq j \leq h-m$ and $k+m \leq j \leq h$, $\bar{j} \in \mathcal{J}(\bar{k}, \bar{h})$, and as pointed out in Remark 7.2, $\gamma_{\bar{j}}$ and $F_{\bar{k}, \bar{j}}(\gamma_{\bar{j}})$ are contained in $X_{\bar{k}, \bar{h}}$. Consequently, for these values of j , $\gamma_j \in H_k(X_{\bar{k}, \bar{h}})$. On the other hand, if $k < j < h$, and j

does not fall into one of the above two cases, then $h - m + 1 \leq j \leq k + m - 1$, which implies $0 \leq j - k, h - j \leq m - 1$ and hence $i(\gamma_j, \gamma_k) = i(\gamma_j, \gamma_h) = 0$, and hence γ_j is disjoint from $H_k(X_{\bar{k}, \bar{h}})$. This completes the proof of the second statement.

When $h - k = 2m - 1$, $X_{\bar{k}, \bar{h}} = S$, and hence $\{\gamma_{\bar{k}}, F_{\bar{k}, \bar{h}}(\gamma_{\bar{h}})\}$ fills S . Consequently $\{\gamma_k, \gamma_h\}$ also fills S .

Now we must prove that for $h - k \geq 2m - 1$, that γ_k and γ_h fill S . The proof is by induction, but we need a little more information in the induction. For simplicity, we assume that $k \geq m + 1$ to avoid special cases.

To describe the additional conditions, for $k < l$, let $\Phi_l = \phi_m \cdots \phi_l$, so that Φ_{k+m-1}^{-1} sends the curves $\gamma_k, \dots, \gamma_h$ (in order) to the curves

$$\gamma_{\bar{k}}, \dots, \gamma_{\overline{k+m-1}}, \phi_{k+m}(\gamma_{\overline{k+m}}), \dots, \phi_{k+m} \cdots \phi_h(\gamma_{\bar{h}}).$$

With this notation, we now wish to prove by double induction (on k and $h - k$) that for all $m + 1 \leq k < h$ with $h - k \geq 2m - 1$ we have

$$\{\gamma_k, \gamma_h\} \text{ fills } S \text{ and } d_{\Phi_{k+m-1}(X_{\bar{k}})}(\gamma_k, \gamma_h) \geq 12.$$

The base case is $h - k = 2m - 1$ and any $k \geq m + 1$. We have already pointed out that $\{\gamma_k, \gamma_h\}$ fills S . We note that applying Φ_{k+m}^{-1} takes $\gamma_{k+1}, \dots, \gamma_h$ to

$$\gamma_{\overline{k+1}}, \dots, \gamma_{\overline{k+m}}, \phi_{k+m+1}(\gamma_{\overline{k+m+1}}), \dots, \phi_{k+m} \cdots \phi_{k+2m-1}(\gamma_{\overline{k+2m-1}}).$$

For the first and last curves $\{\gamma_{\overline{k+1}}, \phi_{k+m+1} \cdots \phi_{k+2m-1}(\gamma_{\overline{k+2m-1}})\}$ we see that these fill $X_{\overline{k+1}, \overline{k+2m-1}} = X_{\overline{k+1}, \overline{k-1}}$ which has $\gamma_{\bar{k}}$ as a boundary component. Since $\pi_{\bar{k}}(\phi_{k+m+1} \cdots \phi_{k+2m-1}(\gamma_{\overline{k+2m-1}}))$ is disjoint from $\gamma_{\bar{k}}$ it follows that applying ϕ_{k+m} to this last curve $\phi_{k+m+1} \cdots \phi_{k+2m-1}(\gamma_{\overline{k+2m-1}})$ we have

$$d_{\bar{k}}(\gamma_{\bar{k}}, \phi_{k+m} \phi_{k+m+1} \cdots \phi_{k+2m-1}(\gamma_{\overline{k+2m-1}})) \geq 14 > 12.$$

But notice that $\Phi_{k+m-1}^{-1}(\gamma_{k+2m-1}) = \phi_{k+m} \cdots \phi_{k+2m-1}(\gamma_{\overline{k+2m-1}})$ while on the other hand $\Phi_{k+m-1}^{-1}(\gamma_k) = \gamma_{\bar{k}}$, hence

$$d_{\Phi_{k+m-1}(X_{\bar{k}})}(\gamma_k, \gamma_{k+2m-1}) \geq 12,$$

as required for the base case.

For the induction step, the proof is quite similar. We assume that the statement holds for all $k \geq m + 1$ and all $2m - 1 \leq h - k \leq N$, and prove it for $h - k = N + 1$. Since $h - (k + 1) = N$ and $k + 1 \geq m + 2 \geq m + 1$, by the inductive assumption it follows that $\{\gamma_{k+1}, \gamma_h\}$ fills S and that

$$d_{\Phi_{k+m}(X_{\overline{k+1}})}(\gamma_{k+1}, \gamma_h) \geq 12.$$

Therefore, applying Φ_{k+m}^{-1} , we have

$$d_{\overline{k+1}}(\gamma_{\overline{k+1}}, \phi_{k+m+1} \cdots \phi_h(\gamma_{\bar{h}})) \geq 12.$$

The homeomorphism Φ_{k+m}^{-1} sends $\gamma_k, \dots, \gamma_h$ to the sequence

$$\phi_{k+m}^{-1}(\gamma_{\bar{k}}), \gamma_{\overline{k+1}}, \dots, \gamma_{\overline{k+m}}, \phi_{k+m+1}(\gamma_{\overline{k+1}}), \dots, \phi_{k+m+1} \cdots \phi_h(\gamma_{\bar{h}}).$$

Since $\gamma_{\bar{k}} \subset \partial X_{\bar{k}+1}$ and $\gamma_{\bar{k}+1} \subset \partial X_{\bar{k}}$, Theorem 2.9 (see also Remark 2.10) ensures that we have

$$d_{\bar{k}}(\gamma_{\bar{k}}, \phi_{k+m+1} \cdots \phi_h(\gamma_{\bar{h}})) \leq 4.$$

Applying ϕ_{k+m} (which translates by at least 16 on $\mathcal{C}(X_{\bar{k}})$) to the second curve we get

$$d_{\bar{k}}(\gamma_{\bar{k}}, \phi_{k+m} \phi_{k+m+1} \cdots \phi_h(\gamma_{\bar{h}})) \geq 12.$$

In particular, we have

$$d_{\Phi_{k+m-1}(X_{\bar{k}})}(\gamma_k, \gamma_h) \geq 12.$$

This proves part of the requirement on γ_k, γ_h .

We must also show that $\{\gamma_k, \gamma_h\}$ fills S . We will show that the Φ_{k+m-1} -image, $\{\gamma_{\bar{k}}, \phi_{k+m} \cdots \phi_h(\gamma_{\bar{h}})\}$ fills S , which will suffice. To see this, take any essential curve δ and suppose it is disjoint from both $\gamma_{\bar{k}}$ and $\phi_{k+m} \cdots \phi_h(\gamma_{\bar{h}})$. Then note that δ must have empty projection to $X_{\bar{k}}$, for otherwise the triangle inequality implies that the distance from $\pi_{\bar{k}}(\gamma_{\bar{k}})$ to $\pi_{\bar{k}}(\phi_{k+m} \cdots \phi_h(\gamma_{\bar{h}}))$ is at most 4, a contradiction to the fact that

$$d_{\bar{k}}(\gamma_{\bar{k}}, \phi_{k+m+1} \cdots \phi_h(\gamma_{\bar{h}})) = d_{\Phi_{k+m-1}(X_{\bar{k}})}(\gamma_k, \gamma_h) \geq 12.$$

Since $\{\gamma_{\bar{k}+1}, \phi_{k+m+1} \cdots \phi_h(\gamma_{\bar{h}})\}$ fills S , δ must intersect one of these curves. However, $\gamma_{\bar{k}+1}$ is contained in the boundary of $X_{\bar{k}}$, and hence δ is disjoint from this. Consequently, δ must intersect $\phi_{k+m+1} \cdots \phi_h(\gamma_{\bar{h}})$. Since ϕ_{k+m} is supported on $X_{\bar{k}}$ which is disjoint from δ we have

$$\begin{aligned} 0 \neq i(\delta, \phi_{k+m+1} \cdots \phi_h(\gamma_{\bar{h}})) &= i(\phi_{k+m}^{-1}(\delta), \phi_{k+m+1} \cdots \phi_h(\gamma_{\bar{h}})) \\ &= i(\delta, \phi_{k+m} \cdots \phi_h(\gamma_{\bar{h}})). \end{aligned}$$

This contradicts our initial assumption on δ , and hence no such δ exists and $\{\gamma_{\bar{h}}, \phi_{k+m} \cdots \phi_h(\gamma_{\bar{h}})\}$ fills S as required. This completes the proof. \square

7.3. Specific examples. Here we provide two specific families of examples of the general construction, but it is quite flexible and easy to build many more examples. We need to describe $\gamma_0, \dots, \gamma_{m-1}$, together with the rest of the data from the beginning of Section 7.1. For this, we will first ensure that all of our subsurfaces X_k have the property that $\gamma_{k\pm 1} \subseteq \partial X_k$ (indices mod m). This is the first of the four conditions required. For the other three conditions, it will be enough to choose the sequence so that for any $0 \leq k, h \leq m-1$, there is a homeomorphism of pairs $(X_k, \gamma_k) \cong (X_h, \gamma_h)$. For then, we can choose $f_0: S \rightarrow S$ any homeomorphism which is the identity on $S \setminus X_k$, pseudo-Anosov on $\mathcal{AC}(X_k)$ with translation distance at least 15, and then use the homeomorphisms $(X_0, \gamma_0) \cong (X_k, \gamma_k)$ to conjugate f_0 to homeomorphisms $f_k: S \rightarrow S$.

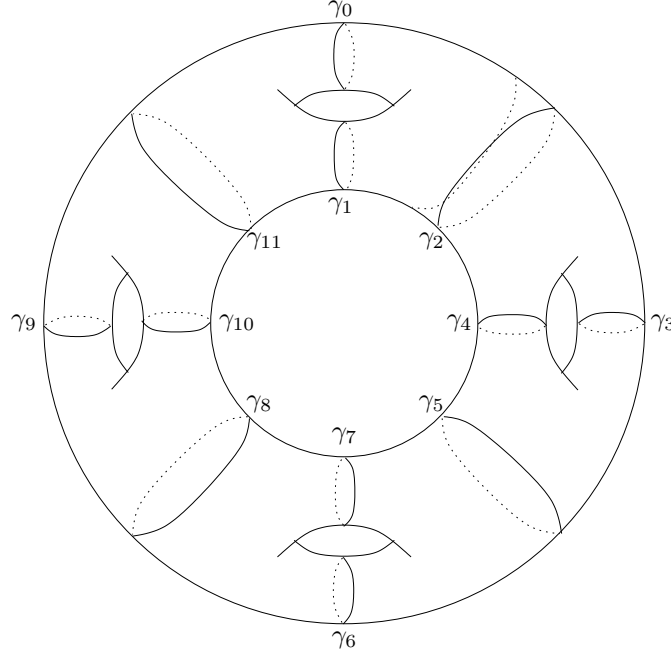


FIGURE 1. The pairwise disjoint curves $\gamma_0, \dots, \gamma_{m-1}$ for the first family of examples in the case of genus 5 (and hence $m = 12$).

7.3.1. *Maximal dimensional simplices.* For the first family of examples, we can choose a pants decomposition on $S_{g,0}$ a closed genus $g \geq 3$ surface as shown in Figure 1. Each X_k is homeomorphic to a 4-holed sphere, and $\gamma_k \subset X_k$ is an essential curve. Any two (X_k, γ_k) and (X_h, γ_h) are clearly homeomorphic pairs. In this case $m = 3g - 3$, and the limiting lamination ν from Proposition 4.4 defines a simplex of measures with maximal possible dimension in $\mathcal{PML}(S)$ by Theorem 6.7. One can also construct examples in genus 2 by taking $\gamma_0, \gamma_1, \gamma_2$ to be a pants decomposition of non-separating curves.

7.3.2. *Non-maximal examples.* For our second family, we choose $m = g - 1$, and take a sequence $\gamma_0, \dots, \gamma_{m-1}$ as shown in Figure 2. Here each X_k is homeomorphic to a surface of genus 2 with two boundary components and γ_k is a curve that cuts X_k into two genus 1 surfaces with two boundary components.

8. TEICHMÜLLER GEODESICS AND ACTIVE INTERVALS

In [Raf05],[Raf07] and [Raf14] the fourth author has developed techniques to control the length-functions and twist parameters along Teichmüller geodesics in terms of subsurface coefficients. In [LLR] this control was used to study the limit sets of Teichmüller geodesics in the Thurston compactification of

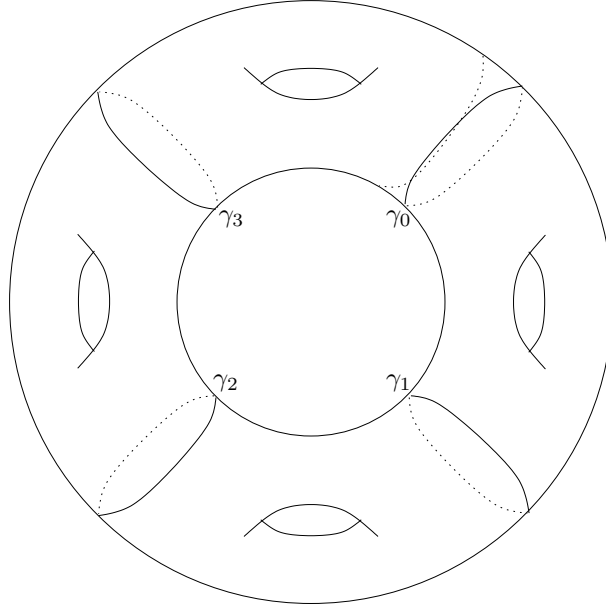


FIGURE 2. The pairwise disjoint curves $\gamma_0, \dots, \gamma_{m-1}$ for the second family in the case of genus 5 (and hence $m = 4$).

Teichmüller space. Here we also appeal to this control. Most of the estimates in this section are similar to the ones in §6 of [LLR].

For the remainder of this section and the next we assume that $\{\gamma_k\}_{k=0}^\infty$ is a sequence of curves satisfying the condition \mathcal{P} from Definition 3.1 with $a > 1$ large enough to satisfy (5.6) and consequently so that (5.3) in Theorem 5.1 holds, and the sequence of powers $\{e_k\}_{k=0}^\infty$ satisfy the growth condition (3.1) for this a . For $h = 0, \dots, m-1$, let $\gamma_i^h = \gamma_{im+h}$, as usual.

Let ν be the nonuniquely ergodic lamination determined by the sequence (see Theorem 4.3 and Corollary 6.2). Furthermore let $\bar{\nu}^h$, for $h = 0, \dots, m-1$, be the ergodic measures from Theorems 5.10 and 6.7, so that $\gamma_i^h \rightarrow \bar{\nu}^h$ in $\mathcal{PML}(S)$, for each h . Let

$$\bar{\nu} = \sum_{h=0}^{m-1} x_h \bar{\nu}^h,$$

for any $x_h > 0$ for each $h = 0, \dots, m-1$.

Let $X \in \text{Teich}(S)$ and μ be a short marking at X . By [HM79], there is a unique Teichmüller geodesic ray starting at X with vertical foliation $\bar{\nu}$, and we let $\bar{\eta}$ be the horizontal foliation (with support η). Denote the Teichmüller geodesic ray by $r : [0, \infty) \rightarrow \text{Teich}(S)$. For a $t \in \mathbb{R}$, we sometimes denote $r(t) = X_t$ and denote the quadratic differential at X_t by q_t . We write $v_t(\alpha), h_t(\alpha), \ell_t(\alpha)$ for the q_t -vertical variation, q_t -horizontal variation, and q_t -length of α , respectively. In particular, $v_t(\alpha) = \exp(-t)i(\alpha, \bar{\eta})$, $h_t(\alpha) = \exp(t)i(\alpha, \bar{\nu})$, and $\ell_t(\alpha) \stackrel{*}{\asymp} v_t(\alpha) + h_t(\alpha)$.

We write $\text{Hyp}_t(\alpha) = \text{Hyp}_{X_t}(\alpha)$, the X_t -hyperbolic length of α and $w_t(\alpha) = w_{X_t}(\alpha)$ for the X_t -width, and recall from (2.2) that

$$w_t(\alpha) \stackrel{+}{\asymp} 2 \log \left(\frac{1}{\text{Hyp}_t(\alpha)} \right).$$

We also recall that $\epsilon_0 > 0$ is the Margulis constant, and that any two hyperbolic geodesics of length at most ϵ_0 must be embedded and disjoint.

For any curve α let $\text{cyl}_t(\alpha)$ be the maximal flat cylinder foliated by all geodesic representatives of α in the q_t metric, as in §6.1, and let $\text{mod}(\text{cyl}_t(\alpha))$ denote its modulus. Fix $M > 0$ sufficiently large so that for any curve α with $\text{mod}(\text{cyl}_t(\alpha)) \geq M$, for some $t \in \mathbb{R}$, then $\text{Hyp}_t(\alpha) \leq \epsilon_0$. For any $k \in \mathbb{N}$, let J_{γ_k} , also denoted J_k , be the *active interval* of γ_k

$$J_k = \{t \in [0, \infty) \mid \text{mod}(\text{cyl}_t(\gamma_k)) \geq M\}.$$

Write $J_k = [a_k, \bar{a}_k]$ and denote the midpoint of J_k by a_k (the balance time of γ_k along the geodesic, i.e. the unique t when $v_t(\gamma_k) = h_t(\gamma_k)$). For each $h \in \{0, \dots, m-1\}$ and $i \geq 0$, we also write $J_{im+h} = J_i^h$, $a_i^h = a_{im+h}$, $\underline{a}_i^h = \underline{a}_{im+h}$, and $\bar{a}_i^h = \bar{a}_{im+h}$, to denote the data associated to $\gamma_i^h = \gamma_{im+h}$.

Proposition 8.1. (Active intervals of curves in the sequence) *With the assumptions and notation as above, we have the following.*

- (i) For k sufficiently large, $J_k \neq \emptyset$. Moreover $J_k \cap J_l = \emptyset$ whenever $i(\gamma_k, \gamma_l) \neq 0$.
- (ii) For $0 \leq f < k$ sufficiently large with $k - f \geq m$, J_f occurs before J_k . Consequently, some tail of each subsequence $\{J_i^h\}_{i=0}^\infty$ appears in order.
- (iii) For k sufficiently large and a multiplicative constant depending only on ν and X

$$\text{Hyp}_{a_k}(\gamma_k) \stackrel{*}{\asymp} \frac{1}{d_{\gamma_k}(\mu, \nu)} \stackrel{*}{\asymp} \frac{1}{e_k}.$$

- (iv) For an additive constant depending only on ν , X , and M , we have

$$|J_k| \stackrel{\perp}{\asymp} \log d_{\gamma_k}(\mu, \nu) \stackrel{\perp}{\asymp} \log(e_k).$$

The following will be convenient for the proof of Proposition 8.1.

Lemma 8.2. *With notation and assumptions above, there exists $k_0 \geq 0$ sufficiently large so that if $Y \subseteq S$ is a subsurface such that for some $k \geq k_0$, $d(\gamma_k, \partial Y) \leq 2$, then*

$$d_Y(\mu, \nu) \stackrel{\perp}{\asymp}_{G+1} d_Y(\eta, \nu).$$

where G is the constant from Theorem 2.11 (for a geodesic).

Proof. Let g be a geodesic in $\mathcal{C}(S)$ from (any curve in) μ limiting to η if η is an ending lamination, or from μ to any curve α with $i(\alpha, \bar{\eta}) = 0$ otherwise. Since η and ν fill S , and $\gamma_k \rightarrow \nu \in \partial\mathcal{C}(S)$, the distance from γ_k to g tends to infinity with k . For Y and γ_k as in the statement of the lemma, $d_S(\partial Y, \gamma_k) \leq 2$, and hence for k sufficiently large, ∂Y has distance at least 4 from g . Consequently, ∂Y intersects every curve on g , and Theorem 2.11

guarantees that $\text{diam}_Y(g) \leq G$. Thus for all $\beta \in g$, $d_Y(\beta, \mu) \leq G$. Since g limits to η (or one of it is curves is disjoint from η), it follows that $d_Y(\eta, \mu) \leq G + 1$, and so the lemma follows from the triangle inequality in $\mathcal{C}(Y)$. \square

Proof of Proposition 8.1. From [Raf05], if $d_{\gamma_k}(\eta, \nu)$ is sufficiently large, then at the balance time a_k , $\text{cyl}_{a_k}(\gamma_k)$ has modulus at least M . For all k sufficiently large, Proposition 4.5(4.7) and Lemma 8.2 imply

$$d_{\gamma_k}(\eta, \nu) \stackrel{\pm}{\succ} d_{\gamma_k}(\mu, \nu) \stackrel{\pm}{\succ} e_k.$$

By construction, $e_k \rightarrow \infty$ as $k \rightarrow \infty$, and hence $J_k \neq \emptyset$ for all sufficiently large k . Furthermore, for all $t \in J_k$, we have $\text{Hyp}_t(\gamma_k) \leq \epsilon_0$. Since two curves with length bounded by ϵ_0 are disjoint, part (i) follows.

By Proposition 4.5(4.5) we have

$$d_{\gamma_k}(\gamma_f, \gamma_l) \stackrel{\pm}{\succ} e_k$$

for all $0 \leq f < k < l$ with $l - k, k - f \geq m$. Let $K \geq 0$ be such that for all $k \geq K$, $e_k > B_0$, where B_0 is the constant from Proposition 2.9. Thus for all $K < f < k < l$ with $l - k, k - f \geq m$ we have

$$d_{\gamma_f}(\gamma_k, \gamma_l) \leq B_0.$$

Since $\gamma_k \rightarrow \nu \in \partial\mathcal{C}(S)$, the triangle inequality in $\mathcal{C}(\gamma_k)$ implies that

$$d_{\gamma_f}(\gamma_k, \nu) \stackrel{\pm}{\succ} 0$$

for all $K \leq f < k$ with $k - f \geq m$. Let $K_0 \geq K$ be sufficiently large so that if $f \geq K_0$ then $d_{\gamma_f}(\eta, \nu) \stackrel{\pm}{\succ} e_f$. Thus, for $k - f \geq m$, $f \geq K_0$, at the balance time $t = a_f$ of γ_f , the q_t -geodesic representative of γ_k is more vertical than horizontal, and hence $a_f < a_k$. By part (i), the intervals J_f and J_k are disjoint, so part (ii) holds. (See also the discussion in Proposition 5.6 of [Raf05].)

For part (iv), observe that by [Raf05], the modulus of $\text{cyl}_t(\gamma_k)$ satisfies

$$(8.1) \quad \text{mod}(\text{cyl}_t(\gamma_k)) \stackrel{*}{\succ} \frac{d_{\gamma_k}(\eta, \nu)}{\cosh^2(t - a_k)}$$

For k is sufficiently large, Lemma 8.2 implies $d_{\gamma_k}(\eta, \nu) \stackrel{\pm}{\succ} d_{\gamma_k}(\mu, \nu) \stackrel{\pm}{\succ} e_k$. At the endpoint \bar{a}_k of J_k , $\text{mod}(\text{cyl}_{\bar{a}_k}(\gamma_k)) = M$. Since $|J_k| = 2(\bar{a}_k - a_k)$, we have

$$M \stackrel{*}{\succ} \frac{e_k}{\cosh^2(\frac{1}{2}|J_k|)}.$$

Taking logarithms we obtain $\log(e_k) - |J_k| \stackrel{\pm}{\succ} \log(M)$, proving part (iv).

We proceed to the proof of part (iii). Following Rafi in §6 of [Raf05], we introduce the following constants associated to a curve $\alpha \in \mathcal{C}(S)$ and an essential subsurface $Y \subseteq S$ with $\alpha \subseteq \partial Y$ (when Y is an annulus, recall that $\alpha \subseteq \partial Y$ means that α is the core curve of Y).

- If Y is a non-annular subsurface, an arc β in Y is a *common K -quasi-parallel* of $\pi_Y(\eta)$ and $\pi_Y(\nu)$ for α and Y if β transversely intersects α and

$$\max\{i(\beta, \pi_Y(\eta)), i(\beta, \pi_Y(\nu))\} \leq K.$$

Here $\pi_Y(\eta)$ denotes the arc-and-curve projection of η : the union of arcs and curves obtained by intersecting η with Y (likewise for ν). Define $K(Y) = \log K$, where K is the smallest number so that η and ν have a common K -quasi-parallel.

- If Y is an annular subsurface, let $K(Y) = d_Y(\eta, \nu)$.

Now define K_α to be the largest $K(Y)$ where $\alpha \subseteq \partial Y$. Then Theorem 6.1 of [Raf05] implies that $\text{Hyp}_a(\alpha) \stackrel{*}{\asymp} \frac{1}{K_\alpha}$, where a is the balance time of α along the geodesic ray r .

In what follows we show that for all sufficiently large k , K_{γ_k} is approximately equal to e_k . Since we will be interested in subsurfaces Y with $\gamma_k \subseteq \partial Y$ (or subsurfaces of those, $Z \subseteq Y$), we can apply to Lemma 8.2 deducing that $d_Y(\eta, \nu) \stackrel{\pm}{\asymp} d_Y(\mu, \nu)$. We will assume that k is sufficiently large for this to hold, and will use this without further mention.

First suppose Y is the annulus with core curve γ_k , and observe that by Proposition 4.5 and Lemma 8.2, $d_Y(\eta, \nu) \stackrel{\pm}{\asymp} d_Y(\mu, \nu) \stackrel{\pm}{\asymp} e_k$, thus $K(Y) \stackrel{\pm}{\asymp} e_k$. So we consider the case that Y is a non-annular subsurface with $\gamma_k \subseteq \partial Y$, and prove that for sufficiently large k , $K(Y) \prec e_k$.

If Y contains no curves γ_k from the sequence as essential curves, then for every subsurface $Z \subseteq Y$, by Proposition 4.5 and Lemma 8.2 we have $d_Z(\eta, \nu) \stackrel{\pm}{\asymp} d_Z(\mu, \nu) \stackrel{\pm}{\asymp} 0$. Then choosing the threshold A in Theorem 2.7 larger than the upper bound on these projections, and applying the theorem to $\pi_Y(\eta), \pi_Y(\nu)$, we see that $i(\pi_Y(\eta), \pi_Y(\nu)) \stackrel{\pm}{\asymp} 0$. In this case we have $K(Y) \stackrel{\pm}{\asymp} 0$, and so $K(Y) \prec e_k$ for all sufficiently large k .

Next we suppose there are curves from our sequence contained in Y . Let $\{\gamma_l\}_{l \in \mathcal{L}} \subseteq \{\gamma_f\}_{f=0}^\infty$ where \mathcal{L} is an ordered subset of \mathbb{N} which is the set of curves from our sequence which are contained in Y . From (4.1) in Theorem 4.1 we see that $\mathcal{L} \subseteq \{k - m + 1, \dots, k + m - 1\}$ since any other curve in the sequence intersects γ_k . We proceed to find an upper bound for the factor $K(Y)$. For this purpose let $\beta \subseteq \pi_Y(\gamma_{k+m})$ be any component arc of the projection. Then from Theorem 2.7 and Lemma 8.2 we have

$$i(\beta, \pi_Y \nu) \asymp \sum_{\substack{W \subseteq Y, \\ \text{non-annular}}} \{d_W(\gamma_{k+m}, \nu)\}_A + \sum_{\substack{W \subseteq Y, \\ \text{annular}}} \log \{d_W(\gamma_{k+m}, \nu)\}_A.$$

and

$$\begin{aligned} i(\beta, \pi_Y \eta) &\asymp \sum_{\substack{W \subseteq Y, \\ \text{non-annular}}} \{d_W(\gamma_{k+m}, \eta)\}_A + \sum_{\substack{W \subseteq Y, \\ \text{annular}}} \log \{d_W(\gamma_{k+m}, \eta)\}_A \\ &\asymp \sum_{\substack{W \subseteq Y, \\ \text{non-annular}}} \{d_W(\gamma_{k+m}, \mu)\}_A + \sum_{\substack{W \subseteq Y, \\ \text{annular}}} \log \{d_W(\gamma_{k+m}, \mu)\}_A. \end{aligned}$$

Choose the threshold constant A from Theorem 2.7 larger than the constant $R(\mu)$ from Proposition 4.5. Appealing to that proposition and the fact that any $l \in \mathcal{L}$ is less than $k + m$, the first of these equations implies that $i(\beta, \pi_Y \nu) \asymp 0$. For the second set of equations, note that any $l \in \mathcal{L}$ with $\gamma_l \cap \gamma_{k+m}$ has $l \leq k$. Therefore, by Theorem 2.7 and the fact that $\{e_f\}$ is increasing, we have

$$\begin{aligned} i(\beta, \pi_Y \mu) &\asymp \sum_{l \in \mathcal{L}} \log \{d_{\gamma_l}(\gamma_{k+m}, \mu)\}_A \prec \sum_{l=k-m+1}^k \log(d_{\gamma_l}(\gamma_{k+m}, \mu)) \\ &\asymp \sum_{l=k-m+1}^k \log(e_l) \prec m \log(e_k) \prec e_k. \end{aligned}$$

Therefore, β is a K -quasi-parallel with $K \prec e_k$. Consequently, $K(Y) \leq \log(K) \prec \log(e_k) \prec e_k$. This completes the proof of part (iii), and hence the proposition. \square

Next we list some estimates for the locations of the intervals $J_i^h \subseteq [0, \infty)$, and provide more information on the relative positions of the intervals.

Let $h \in \{0, \dots, m-1\}$. From part (i) and (iv) of Proposition 8.1, together with the definitions, we have that for i sufficiently large

$$(8.2) \quad \underline{a}_i^h \stackrel{\pm}{\asymp} a_i^h - \frac{\log e_i^h}{2} \quad \text{and}$$

$$(8.3) \quad \bar{a}_i^h \stackrel{\pm}{\asymp} a_i^h + \frac{\log e_i^h}{2}.$$

Together with these estimates, the next lemma tells us the location of the active intervals, up to an additive error.

Lemma 8.3. *For any $h = \{0, \dots, m-1\}$ and i sufficiently large*

$$(8.4) \quad a_i^h \stackrel{\pm}{\asymp} \sum_{j=0}^{i-1} \log b e_j^h + \frac{\log e_i^h}{2} - \frac{\log x_h}{2}.$$

The additive error depends on X , γ_0^h , and ν .

Proof. The proof of this lemma is similar to that of [LLR, Lemma 6.3], so we just sketch the proof. Choose i sufficiently large so that $J_i^h \neq \emptyset$ and $a_i^h > 0$, and so that we may estimate $i(\gamma_i^h, \mu)$ using Lemma 5.11 (since μ is

a finite set of curves). Then appealing to the fact that X is a fixed surface and μ a short marking, we have

$$(8.5) \quad v_0(\gamma_i^h) \stackrel{*}{\asymp} l_0(\gamma_i^h) \stackrel{*}{\asymp} \text{Hyp}_0(\gamma_i^h) \stackrel{*}{\asymp} i(\gamma_i^h, \mu) \stackrel{*}{\asymp} A(0, h + im) = \prod_{j=0}^{i-1} be_j^h.$$

Since $v_t(\gamma_i^h)h_t(\gamma_i^h)$ is constant in t , and $v_{a_i^h}(\gamma_i^h) = h_{a_i^h}(\gamma_i^h)$, we have, for i sufficiently large

$$\begin{aligned} v_{a_i^h}^2(\gamma_i^h) &= v_{a_i^h}(\gamma_i^h)h_{a_i^h}(\gamma_i^h) \\ &= v_0(\gamma_i^h)h_0(\gamma_i^h) \\ &\stackrel{*}{\asymp} i(\gamma_i^h, \mu)i(\gamma_i^h, \bar{\nu}) \\ &\stackrel{*}{\asymp} i(\gamma_i^h, \mu) \left(\sum_{d=0}^{m-1} x_d i(\gamma_i^h, \bar{\nu}^d) \right) \end{aligned}$$

Since μ is a fixed set of curves and γ_0^h a fixed curve, $i(\gamma_0^h, \gamma_i^h) \stackrel{*}{\asymp} i(\mu, \gamma_i^h)$ for all i sufficiently large. Thus from (6.1), for $h \neq d$, $d \in \{0, \dots, m-1\}$, we have

$$i(\gamma_i^h, \bar{\nu}^h) \stackrel{*}{\asymp} \frac{1}{i(\gamma_{i+1}^h, \mu)} \quad \text{and} \quad i(\gamma_i^h, \bar{\nu}^d)i(\gamma_{i+1}^h, \mu) \rightarrow 0$$

The above estimates and Lemma 5.11 imply that for i sufficiently large

$$v_{a_i^h}^2(\gamma_i^h) \stackrel{*}{\asymp} x_h \frac{i(\gamma_i^h, \mu)}{i(\gamma_{i+1}^h, \mu)} \stackrel{*}{\asymp} \frac{x_h}{be_i^h}.$$

Combining this with (8.5) we have

$$\exp(a_i^h) = \frac{v_0(\gamma_i^h)}{\exp(-a_i^h)v_0(\gamma_i^h)} = \frac{v_0(\gamma_i^h)}{v_{a_i^h}(\gamma_i^h)} \stackrel{*}{\asymp} \frac{\prod_{j=0}^{i-1} be_j^h}{\sqrt{x_h/be_i^h}}.$$

Solving for a_i^h and taking logarithms (discarding a constant $\log b$) proves (8.4), completing the proof. \square

Lemma 8.4. *For any k sufficiently large, $\bar{a}_k \stackrel{\pm}{\asymp} \underline{a}_{k+m}$, with additive error depends on X , M , γ_0^h , and ν .*

Proof. Let $k = im + h$ where $h \in \{0, \dots, m-1\}$. From (8.2), (8.3) and (8.4) we calculate

$$\begin{aligned} \underline{a}_{k+m} - \bar{a}_k &= \underline{a}_{i+1}^h - \bar{a}_i^h \\ &\stackrel{\pm}{\asymp} \sum_{j=0}^i \log be_j^h + \frac{\log e_{i+1}^h}{2} - \frac{\log x_h}{2} - \frac{\log e_{i+1}^h}{2} \\ &\quad - \left(\sum_{j=0}^{i-1} \log be_j^h + \frac{\log e_i^h}{2} - \frac{\log x_h}{2} + \frac{\log e_i^h}{2} \right) \\ &= \log be_i^h - \log e_i^h = \log b. \end{aligned}$$

Therefore $\bar{a}_k \stackrel{+}{\asymp} \underline{a}_{k+m}$ since $\log b$ is a constant. \square

Let $k, l \in \mathbb{N}$ and $0 < l - k \leq m$. Suppose that $k \equiv h \pmod{m}$ and $l \equiv d \pmod{m}$ where $h, d \in \{0, \dots, m-1\}$. Then for the pair (k, l) one of the following two hold:

$$(8.6) \quad h < d \text{ and } \exists i \in \mathbb{N}, \text{ so that } k = mi + h \text{ and } l = mi + d, \text{ or}$$

$$(8.7) \quad h > d \text{ and } \exists i \in \mathbb{N}, \text{ so that } k = mi + h \text{ and } l = m(i+1) + d.$$

Notation 8.5. Let $\{x_i\}_{i=0}^\infty$ and $\{y_i\}_{i=0}^\infty$ be sequences of real numbers. We write $x_i \ll y_i$ if $x_i < y_i$ for all i sufficiently large and $y_i - x_i \rightarrow \infty$ as $i \rightarrow \infty$.

Lemma 8.6. For $k, l \in \mathbb{N}$ sufficiently large where $0 \leq l - k < m$ the following holds:

$$(8.8) \quad \bar{a}_{k-m} < \underline{a}_l \ll \bar{a}_k.$$

Proof. The proof is similar to the proof of Lemma 7.3 of [LLR]. For the first inequality, note that $l - (k - m) \geq m$. By Proposition 8.1 parts (i) and (ii), J_{k-m} occurs before J_l , and so we have $\bar{a}_{k-m} < \underline{a}_l$.

Now we show that $\underline{a}_l \ll \bar{a}_k$. If $l = k$, then since $|J_k| \rightarrow \infty$ as $k \rightarrow \infty$, we have $\underline{a}_k \ll \bar{a}_k$. Now assume $k < l$ and let $k \equiv h \pmod{m}$ and $l \equiv d \pmod{m}$ with $h, d \in \{0, \dots, m-1\}$. First, suppose that (8.6) holds so $h < d$. Using (3.1), (8.2), (8.3) and (8.4), and the fact that $e_k \geq a^{k-f} e_f$ for $k > f$, we have

$$\begin{aligned} \bar{a}_k - \underline{a}_l &= \bar{a}_i^h - \underline{a}_i^d \\ &\stackrel{+}{\asymp} \sum_{j=0}^{i-1} \log be_j^h + \log e_i^h - \frac{1}{2} \log x_h - \sum_{j=0}^{i-1} \log be_j^d + \frac{1}{2} \log x_d \\ &= \sum_{j=0}^{i-1} \log \frac{e_j^h}{e_j^d} + \log e_i^h + \frac{1}{2} \log \frac{x_d}{x_h} \\ &= \sum_{j=1}^i \log \frac{e_j^h}{e_{j-1}^d} + \log e_0^h + \frac{1}{2} \log \frac{x_d}{x_h} \\ &\geq \sum_{j=1}^i (m + h - d) \log a + \frac{1}{2} \log \frac{x_d}{x_h} \\ &= i(m + h - d) \log a + \frac{1}{2} \log \frac{x_d}{x_h}. \end{aligned}$$

Now since $m + h - d > 0$, the last term goes to ∞ as $i \rightarrow \infty$.

Next suppose that (8.7) holds so $h > d$. Then we similarly have

$$\begin{aligned}
\bar{a}_k - \underline{a}_l &= \bar{a}_i^h - \underline{a}_{i+1}^d \\
&\stackrel{(+)}{>} \sum_{j=1}^{i-1} \log be_j^h + \log e_i^h - \sum_{j=1}^i \log be_j^d + \frac{1}{2} \log \frac{x_d}{x_h} \\
&= \sum_{j=1}^i \log \frac{e_j^h}{e_j^d} + \frac{1}{2} \log \frac{x_d}{x_h} - \log b \\
&= \sum_{j=1}^i \log \frac{e_j^h}{e_j^d} + \frac{1}{2} \log \frac{x_d}{x_h} - \log b \\
&= i(h-d) \log a + \frac{1}{2} \log \frac{x_d}{x_h} - \log b.
\end{aligned}$$

Now since $h-d > 0$, the last term goes to ∞ as $i \rightarrow \infty$. \square

To obtain a greater control over the arrangement of intervals J_k along the Teichmüller geodesic ray (see Lemma 8.8 below) we consider the following growth conditions, in addition to (3.1):

$$(8.9) \quad e_{k+1} \geq \left(\prod_{j=0}^k e_j \right)^2.$$

Such sequences exist simply by setting $e_0 \geq a$ and defining e_k recursively, ensuring at every step that (8.9) is satisfied.

Condition (8.9) has the following consequence.

Lemma 8.7. *Suppose a sequence $\{e_k\}_k$ satisfies (3.1) and (8.9).*

$$(8.10) \quad \begin{aligned} &\text{If (8.6) holds, then } \frac{(e_i^d)^{\frac{1}{2}}}{e_i^h} \prod_{j=0}^{i-1} \frac{e_j^d}{e_j^h} \rightarrow \infty, \text{ and} \\ &\text{If (8.7) holds, then } (e_{i+1}^d)^{\frac{1}{2}} \prod_{j=0}^i \frac{e_j^d}{e_j^h} \rightarrow \infty. \end{aligned}$$

Proof. Let $k \equiv d \pmod{m}$ and $l \equiv h \pmod{m}$, where $d, h \in \{0, \dots, m-1\}$. First suppose that (8.6) holds so $h < d$. Since $\{e_k\}$ is increasing (more than) exponentially fast

$$\prod_{j=0}^{i-1} \frac{e_j^d}{e_j^h} \rightarrow \infty.$$

Moreover, by (8.9) we have $(e_i^d)^{\frac{1}{2}} \geq e_i^h$, that is, $(e_i^d)^{\frac{1}{2}}/e_i^h \geq 1$. Thus (8.10) follows.

Now suppose that (8.7) holds so $h > d$. Then

$$(e_{i+1}^d)^{\frac{1}{2}} \geq \prod_{j=0}^{m(i+1)+d-1} e_j \geq \prod_{j=0}^i e_j^h.$$

where the second inequality holds because $m(i+1) + d > mi + h$. Therefore, condition (8.10) easily follows in this case as well. \square

Lemma 8.8. *Suppose that the growth condition (8.9) holds. Then for $k, l \in \mathbb{N}$ sufficiently large with $0 < l - k < m$ we have*

$$(8.11) \quad \bar{a}_k \ll a_l$$

Proof. Let $f \equiv h \pmod{m}$ and $l \equiv d \pmod{m}$ where $h, d \in \{0, \dots, m-1\}$.

First suppose that (8.6) holds so $h < d$. Then from (8.3) and (8.4) we calculate

$$\begin{aligned} a_l - \bar{a}_k &= a_i^d - \bar{a}_i^h \\ &\stackrel{+}{>} \sum_{j=0}^{i-1} \log be_j^d + \frac{\log e_i^d}{2} - \frac{\log x_d}{2} - \left(\sum_{j=0}^{i-1} \log be_j^h + \log e_i^h - \frac{\log x_h}{2} \right) \\ &= \log \left(\frac{(e_i^d)^{\frac{1}{2}}}{e_i^h} \prod_{j=0}^{i-1} \frac{e_j^d}{e_j^h} \right) + \frac{1}{2} \log \frac{x_h}{x_d} \rightarrow \infty \end{aligned}$$

where the sequence tends to infinity as $i \rightarrow \infty$ by Lemma 8.7.

Now suppose that (8.7) holds so $h > d$. Then we have

$$\begin{aligned} a_l - \bar{a}_k &= a_{i+1}^d - \bar{a}_i^h \\ &\stackrel{+}{>} \sum_{j=0}^i \log be_j^d + \frac{\log e_{i+1}^d}{2} - \frac{\log x_d}{2} - \left(\sum_{j=0}^{i-1} \log be_j^h + \log e_i^h - \frac{\log x_h}{2} \right) \\ &= \log \left((e_{i+1}^d)^{\frac{1}{2}} \prod_{j=0}^i \frac{e_j^d}{e_j^h} \right) + \log b + \frac{1}{2} \log \frac{x_h}{x_d} \rightarrow \infty \end{aligned}$$

where again the convergence to infinity as $i \rightarrow \infty$ is by Lemma 8.7. \square

The following conveniently summarizes the relative positions of intervals for large indices. See Figure 3.

Lemma 8.9. *For $k < l$ sufficiently large and $l < k + m$, we have*

$$\underline{a}_k \ll \underline{a}_l \ll a_k \ll \bar{a}_k < \underline{a}_{k+m} \ll a_l \ll \bar{a}_l < \underline{a}_{l+m} \ll a_{k+m}.$$

Furthermore

$$\bar{a}_k \stackrel{+}{>} \underline{a}_{k+m}.$$

Proof. This is immediate from Lemmas 8.4, 8.6 and 8.8. \square

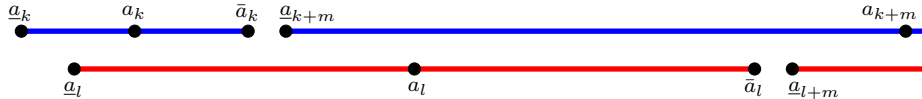


FIGURE 3. Relative positions of active intervals, $k < l < k + m$.

9. LIMIT SETS OF TEICHMÜLLER GEODESICS

In this section, we continue with the assumptions from the previous section on the sequences $\{\gamma_k\}_{k=0}^\infty$ and $\{e_k\}_{k=0}^\infty$ (including both (3.1) and (8.9)), limiting lamination $\nu \in \partial\mathcal{C}(S)$ of $\{\gamma_k\}_{k=0}^\infty$, Teichmüller geodesic ray $r(t) = X_t$ with quadratic differential q_t at time $t \in [0, \infty)$, vertical foliations $\bar{\nu} = \sum_{h=0}^{m-1} x_h \bar{\nu}^h$ and horizontal foliation $\bar{\eta}$ for $(X, q) = (X_0, q_0)$, short marking μ for X , and active intervals $J_k = [\underline{a}_k, \bar{a}_k]$ with midpoint a_k . We will also be appealing to all the estimates from the previous sections regarding this data.

In addition, we will need one more condition on $\{\gamma_k\}_{k=0}^\infty$, which we add to the properties \mathcal{P} assumed already: For any $k \geq 0$, let

$$\sigma_k = \gamma_k \cup \gamma_{k+1} \cup \dots \cup \gamma_{k+m-1}.$$

The additional condition is

$\mathcal{P}(iv)$ Let α be any essential curve in $S \setminus \sigma_k$. Then there is no subsurface $Y \subseteq S$ with $\alpha \subseteq \partial Y$ which is filled by a collection of the curves in the sequence $\{\gamma_k\}_{k=0}^\infty$.

Recall that when Y is an annular subsurface by $\alpha \subseteq \partial Y$ we mean that α is the core curve of Y .

Remark 9.1. Note that when σ_k is a pants decomposition of S the condition $\mathcal{P}(iv)$ holds vacuously because there are no essential curves in $S \setminus \sigma_k$. Together with the other conditions in \mathcal{P} , the new condition $\mathcal{P}(iv)$ is equivalent to requiring that any subsurface filled by a subset of $\{\gamma_k\}_{k=0}^\infty$ has as boundary a union of curves in $\{\gamma_k\}_{k=0}^\infty$. According to Lemma 7.7 condition $\mathcal{P}(iv)$ holds for the sequences constructed in §7.

Under these assumptions, Theorem 1.4 from the introduction, which describes the limit set of $r(t)$ in the Thurston compactification $\overline{\text{Teich}(S)} = \text{Teich}(S) \cup \mathcal{PML}(S)$, can be restated as follows. Recall that the set of projective classes of measures on ν is a simplex $\Delta(\nu)$ spanned by the projective classes of the ergodic measures $[\bar{\nu}^0], \dots, [\bar{\nu}^{m-1}]$.

Theorem 9.2. *The accumulation set of $r(t)$ in $\mathcal{PML}(S)$ is the simple closed curve in the simplex $\Delta(\nu)$ that is the concatenation of edges*

$$[[\bar{\nu}^0], [\bar{\nu}^1]] \cup [[\bar{\nu}^1], [\bar{\nu}^2]] \cup \dots \cup [[\bar{\nu}^{m-1}], [\bar{\nu}^0]].$$

We begin by reducing this theorem to a more manageable statement (Theorem 9.3), which also provides more information about how the sequence limits to the simple closed curve. We then briefly sketch the idea of the proof, and describe some of the necessary estimates. After that we reduce the theorem further to a technical version (Theorem 9.17), providing even more detailed information about what the limit looks like, and which allows for a more concise proof. After supplying the final estimates necessary, we carry out the proof.

9.1. First reduction and sketch of proof. By Proposition 8.1, the intervals J_k are nonempty for all k sufficiently large. Combining this with Lemma 8.8, it follows that for all $k < l$ sufficiently large, $\bar{a}_k < \bar{a}_l$, and that $\bar{a}_l \rightarrow \infty$ with l . Therefore, the set of intervals $[\bar{a}_k, \bar{a}_{k+1}]$ for all sufficiently large k , cover all but a compact subset of $[0, \infty)$, and consecutive segments intersect only in their endpoints. Theorem 9.2 easily follows from

Theorem 9.3. *Fix $h, h' \in \{0, \dots, m-1\}$ with $h' \equiv h+1 \pmod{m}$ and suppose $\{t_i\}$ is a sequence with $t_i \in [\bar{a}_{im+h}, \bar{a}_{im+h+1}]$ for all sufficiently large i . Then $r(t_i) = X_{t_i}$ accumulates on the edge $[[\bar{\nu}^h], [\bar{\nu}^{h'}]] \subset \Delta(\nu)$.*

Furthermore, if $\{t_i - \bar{a}_{im+h}\}$ is bounded independent of i , then

$$\lim_{i \rightarrow \infty} X_{t_i} = [\bar{\nu}^h].$$

Proof of Theorem 9.2 assuming 9.3. From the second part of Theorem 9.3 applied to $t_i = \bar{a}_{im+h}$, it follows that

$$\lim_{i \rightarrow \infty} X_{\bar{a}_{im+h}} = [\bar{\nu}^h].$$

for all $h \in \{0, \dots, m-1\}$. If $h' \equiv h+1$ as in Theorem 9.3, then combining this with the first part of that theorem, we see that the accumulation set of the sequence of subsets $\{r([\bar{a}_{im+h}, \bar{a}_{im+h+1}])\}_{i=0}^{\infty} \subset \text{Teich}(S)$ is contained in $[[\bar{\nu}^h], [\bar{\nu}^{h'}]]$ and contains the endpoints. Consequently, any Hausdorff limit of this sequence of *connected* sets is a connected subset of $[[\bar{\nu}^h], [\bar{\nu}^{h'}]]$ containing the endpoints, and hence is equal to $[[\bar{\nu}^h], [\bar{\nu}^{h'}]]$. The accumulation set of this sequence of sets therefore contains $[[\bar{\nu}^h], [\bar{\nu}^{h'}]]$, and is thus equal to it. Since this holds for every $h \in \{0, \dots, m-1\}$, and the intervals $\{[\bar{a}_k, \bar{a}_{k+1}]\}$ cover all but a compact subset of $[0, \infty)$, this completes the proof. \square

Remark 9.4. Before proceeding we note that the assumptions on $\{\gamma_k\}_{k=0}^{\infty}$ and $\{e_k\}_{k=0}^{\infty}$ are “shift invariant”, meaning that if we start the sequence at any $k_0 \geq 0$, and reindex (without changing the order), the resulting sequence will also satisfy all the required conditions. Consequently, it suffices to prove Theorem 9.3 for $h = 0$ and $h' = 1$. This greatly simplifies the notation, and allows us to avoid duplicating essentially identical arguments.

To sketch the proof, we recall that a sequence $\{Z_i\} \subset \text{Teich}(S)$ converges to $[\bar{\lambda}] \in \mathcal{PML}(S)$ if and only if

$$\lim_{i \rightarrow \infty} \frac{\text{Hyp}_{Z_i}(\delta)}{\text{Hyp}_{Z_i}(\delta')} = \frac{i(\bar{\lambda}, \delta)}{i(\bar{\lambda}, \delta')}$$

for all simple closed curves δ, δ' with $i(\bar{\lambda}, \delta') \neq 0$; see §2. Thus we must provide sufficient control over the hyperbolic lengths of curves and relate these to intersection numbers with measures on ν .

Now the idea of the proof of this theorem is as follows. For any sufficiently large t , we estimate hyperbolic lengths $\text{Hyp}_{X_t}(\delta)$ in terms of “contributions” from the intersections of δ with the curves in a bounded length pants decomposition (Proposition 9.6). When t is in the interval $[\bar{a}_k, \bar{a}_{k+1}]$ we choose a

bounded length pants decomposition containing either σ_k or σ_{k+1} , depending on more precise information about t . The contributions from the curves in these sub-multicurves dominate the contributions from the other curves (the ratios tend to zero), and so the key is to understand these contributions.

On the active interval J_l , the contribution from γ_l grows linearly in the first half of the interval (Lemma 9.10), but during the second half, they speed up. Thus near \bar{a}_k , the contribution from γ_k will be greater than from the rest of σ_k , since \bar{a}_k is still in the first half of J_l , for $l = k+1, \dots, k+m-1$. As we proceed far beyond \bar{a}_k , the bounded length pants decomposition eventually changes to become σ_{k+1} . The contribution from γ_k transitions to the contribution from γ_{k+m} and until the contribution from γ_{k+1} speeds up, this is the dominating term. However, as the contribution from γ_{k+1} speeds up, its contribution eventually takes over. During the transition, the contribution from γ_l , for $2 \leq l \leq m-1$ is still dominated by either γ_{k+m} or γ_{k+1} .

With this sketch in mind, we now start to discuss the details.

9.2. General hyperbolic geometry estimates. For a curve $\alpha \in \mathcal{C}(S)$ and $Z \in \text{Teich}(S)$, we have the length and width $\text{Hyp}_Z(\alpha)$ and $w_Z(\alpha)$, respectively, as defined in §2. Given two curves $\alpha, \delta \in \mathcal{C}(S)$ and $Z \in \text{Teich}(S)$ we will also need the *twist of δ about α with respect to Z* , denoted $\text{tw}_\alpha(\delta, Z)$. This is defined as

$$\text{tw}_\alpha(\delta, Z) = \text{diam}_\alpha(\pi_\alpha(\delta) \cup \alpha^{\perp Z}) \geq 0$$

where $\alpha^{\perp Z}$ is the set of Z -geodesics in the annular cover Y_α meeting (the lift of the geodesic representative of) α orthogonally.

Remark 9.5. There are different definitions of $\text{tw}_\alpha(\delta, Z)$ in the literature (see e.g. [Min96, CR07, CRS08]). Some of these come equipped with a sign which we have no need of, and our definition agrees with (the absolute values of) the other definitions, up to a uniformly bounded additive error (at least those we will be appealing to).

For curves $\alpha, \delta \in \mathcal{C}(S)$ and $Z \in \text{Teich}(S)$ define the *contribution to the Z -length of δ coming from α* by

$$(9.1) \quad \text{Hyp}_Z(\delta, \alpha) := i(\delta, \alpha) \left[w_Z(\alpha) + \text{tw}_\alpha(\delta, Z) \text{Hyp}_Z(\alpha) \right],$$

The next fact, from [CRS08, Lemma 7.2], provides our primary means of control on hyperbolic lengths.

Proposition 9.6. *Given $L > 0$ and $Z \in \text{Teich}(S)$, suppose that P is an L -bounded length pants decomposition ($\text{Hyp}_Z(\alpha) \leq L$ for all $\alpha \in P$). Then for any curve $\delta \in \mathcal{C}(S)$ we have*

$$(9.2) \quad \left| \text{Hyp}_Z(\delta) - \sum_{\alpha \in P} \text{Hyp}_Z(\delta, \alpha) \right| = O\left(\sum_{\alpha \in P} i(\delta, \alpha) \right)$$

where the constant of the O -notation depends only on L .

To effectively use this proposition to analyze lengths of curves in X_t as $t \rightarrow \infty$ we must develop a better picture of the hyperbolic geometry of bounded length curves in X_t .

9.3. Hyperbolic estimates for $\{\gamma_k\}$. As in §8, we will write $\text{Hyp}_t(\alpha) = \text{Hyp}_{X_t}(\alpha)$, $\text{Hyp}_t(\delta, \alpha) = \text{Hyp}_{X_t}(\delta, \alpha)$, and $w_t(\alpha) = w_{X_t}(\alpha)$. By a result of Wolpert [Wol79], hyperbolic lengths change (grow/shrink) at most exponentially in Teichmüller distance, and hence we have

Lemma 9.7. *For any curve α and any $t, s \in \mathbb{R}$, we have*

$$\text{Hyp}_t(\alpha) \leq \exp(2(|t - s|)) \text{Hyp}_s(\alpha).$$

From Lemma 8.9, all sufficiently large t are either contained in exactly m intervals J_k, \dots, J_{k+m-1} or in exactly $m - 1$ intervals $J_{k+1}, \dots, J_{k+m-1}$ and the *bounded length* interval $[\bar{a}_k, \underline{a}_{k+m}]$ (the interval after J_k but before J_{k+m}). In the former case, every curve in σ_k has length at most ϵ_0 , the Margulis constant. In the latter case, we can use Lemma 9.7 to bound the length of curves in σ_k . It will be useful to have a slight generalization of that, which we state here.

Lemma 9.8. *For any $W > 0$, if t is sufficiently large (depending on W), is contained in $J_{k+1}, J_{k+2}, \dots, J_{k+m-1}$, and satisfies $0 < t - \bar{a}_k < W$, then every curve in σ_k has X_t -length at most $\exp(2W)\epsilon_0$.*

Proof. Since \bar{a}_k is in all the intervals J_k, \dots, J_{k+m-1} , we have $\text{Hyp}_{\bar{a}_k}(\gamma_l) \leq \epsilon_0$ for $k \leq l \leq k + m - 1$. Now apply Lemma 9.7. \square

In particular, note that once k is sufficiently large, Lemma 8.4 guarantees that $\underline{a}_{k+m} - \bar{a}_k$ is uniformly bounded by some constant W_0 , and so setting $L_0 = \exp(2W_0)\epsilon_0$, we see that for any sufficiently large t , there is always some k so that all curves of σ_k have length at most L_0 . In addition, this gives us lower bounds on lengths as well.

Lemma 9.9. *For all k sufficiently large, $\text{Hyp}_{\underline{a}_k}(\gamma_k) \stackrel{*}{\asymp} 1 \stackrel{*}{\asymp} \text{Hyp}_{\bar{a}_k}(\gamma_k)$.*

The multiplicative constant here depends only on W_0 , the constants in property \mathcal{P} , and the Margulis constant ϵ_0 .

Proof. We already have $\text{Hyp}_{\underline{a}_k}(\gamma_k) \leq \epsilon_0$, so we need to prove a uniform lower bound. Since $i(\gamma_k, \gamma_{k-m}) \in [b_1, b_2]$ from \mathcal{P} , and $\text{Hyp}_{\underline{a}_k}(\gamma_{k-m}) \leq L_0 = \exp(2W_0)\epsilon_0$, according to Lemma 2.12 we have

$$\text{Hyp}_{\underline{a}_k}(\gamma_k) \geq w_{\underline{a}_k}(\gamma_{k-m})i(\gamma_k, \gamma_{k-m}) \geq 2 \sinh^{-1}(1/\sinh(L_0/2))b_1.$$

A similar argument applies for the estimate on $\text{Hyp}_{\bar{a}_k}(\gamma_k)$. \square

We will also need good estimates on $w_t(\gamma_k)$, especially on the first half of the interval when γ_k initially becomes short.

Lemma 9.10. *For all sufficiently large k and $t \in [\underline{a}_k, a_k]$, we have*

$$w_t(\gamma_k) \stackrel{\pm}{\asymp} 4(t - \underline{a}_k).$$

The implicit constant depends on the constant from Lemma 9.9.

Remark 9.11. There is a mistake in [LLR, Lemma 8.3], which claims that the width grows at most linearly with coefficient 1 (instead of 4). This does not affect any of the proofs. It is also worth noting that only an upper bound was proved there, whereas here there are both upper and lower bounds.

Proof. We first prove the upper bound on $w_t(\gamma_k)$. For this, we note that by Lemma 9.7

$$1 \stackrel{*}{\asymp} \text{Hyp}_{\underline{a}_k}(\gamma_k) \leq \exp(2(t - \underline{a}_k)) \text{Hyp}_t(\gamma_k).$$

Dividing by $\text{Hyp}_t(\gamma_k)$ and taking logarithms, we get

$$\log\left(\frac{1}{\text{Hyp}_t(\gamma_k)}\right) \stackrel{+}{\prec} 2(t - \underline{a}_k).$$

Multiplying by 2 and applying (2.2) proves

$$w_t(\gamma_k) \stackrel{+}{\prec} 4(t - \underline{a}_k).$$

For the lower bound, we will appeal to (8.1), which for k sufficiently large implies

$$\text{mod}(\text{cyl}_t(\gamma_k)) \stackrel{*}{\asymp} \frac{e_k}{\cosh^2(t - a_k)}.$$

Lifting $\text{cyl}_t(\gamma_k)$ to the annular cover Y_{γ_k} , the modulus of the former is bounded above by the modulus of the latter by monotonicity of modulus of annuli. The latter on the other hand can be computed explicitly as $\pi/\text{Hyp}_t(\gamma_k)$ (see e.g. [Mas85]). Thus, taking logs and noting that

$$\log(\cosh^2(t - a_k)) \stackrel{+}{\asymp} 2|t - a_k| = 2(a_k - t),$$

we have

$$\log(e_k) - 2(a_k - t) \stackrel{+}{\prec} \log\left(\frac{\pi}{\text{Hyp}_t(\gamma_k)}\right).$$

Then by Proposition 8.1 we have $\log(e_k) \stackrel{+}{\asymp} 2(a_k - \underline{a}_k)$ and hence

$$2(t - \underline{a}_k) \stackrel{+}{\prec} \log\left(\frac{1}{\text{Hyp}_t(\gamma_k)}\right).$$

Appealing to (2.2) again we have

$$4(t - \underline{a}_k) \stackrel{+}{\prec} w_t(\gamma_k).$$

□

We will also want to estimate $\text{tw}_{\gamma_k}(\delta, X_t)$, for an arbitrary curve δ . This is given by the following formula from [Raf07].

Theorem 9.12. *Given a curve $\delta \in \mathcal{C}_0(S)$ and large enough $k \in \mathbb{N}$ we have*

$$\text{tw}_{\gamma_k}(\delta, X_t) = \begin{cases} 0 \pm O\left(\frac{1}{\text{Hyp}_{X_t}(\gamma_k)}\right), & t \leq a_k \\ e_k \pm O\left(\frac{1}{\text{Hyp}_{X_t}(\gamma_k)}\right), & t \geq a_k \end{cases}$$

This Theorem shows, in particular, that the twisting is independent of δ (up to an error). In fact, arguing as in Lemma 8.2, we can easily prove that this is the case in general.

Lemma 9.13. *For any two curves δ, δ' and constant L , there exists $T > 0$ with the following property. If $\alpha \in \mathcal{C}(S)$ is a curve and $t_0 \geq T$ with $\text{Hyp}_{t_0}(\alpha) \leq L$, then for all t ,*

$$\text{tw}_\alpha(\delta, X_t) \stackrel{\pm}{\asymp}_G \text{tw}_\alpha(\delta', X_t)$$

where G is the constant from Theorem 2.11 (for geodesics).

Proof. For sufficiently large t_0 , a curve α with bounded length must have bounded distance from some γ_k in $\mathcal{C}(S)$. As in the proof of Lemma 8.2, this can be assumed to be very far from the geodesic in $\mathcal{C}(S)$ between δ and δ' (by assuming t_0 , and hence k , is very large). Appealing to Theorem 2.11, we see that $d_\alpha(\delta, \delta') \leq G$. Since $\text{tw}_\alpha(\delta, X_t)$ is defined in terms of distance in $\mathcal{C}(\alpha)$, the lemma follows from the triangle inequality in $\mathcal{C}(\alpha)$. \square

9.4. Bounded length pants decompositions. When $m = \xi(S)$, then for all sufficiently large times t , there exists k so that σ_k is a bounded length pants decomposition for X_t . In this case, the estimates from the previous section then provide many of the necessary ingredients to apply Proposition 9.6 to control $\text{Hyp}_t(\delta)$, for an arbitrary curve δ .

If $m < \xi(S)$, then a bounded length pants decompositions will contain other curves not in the sequence $\{\gamma_k\}$, and in this section, we describe the necessary estimates to handle the contribution to length from these. The reader only interested in the case $m = \xi(S)$ may skip this subsection.

We begin by bounding from below the length of the other curves in a bounded length pants decomposition.

Lemma 9.14. *There exists $\epsilon > 0$ depending on $R(\mu)$ from Proposition 4.5 such that for all sufficiently large t , if $\text{Hyp}_t(\alpha) \leq \epsilon$, then $\alpha \in \{\gamma_k\}_{k=0}^\infty$.*

Proof. Let α be a curve not in $\{\gamma_k\}_{k=0}^\infty$. We will show that K_α is uniformly bounded. This requires us to bound $K(Z)$ for all essential subsurfaces Z with $\alpha \subseteq \partial Z$; see the proof of Proposition 8.1 for the definition of K_α and $K(Z)$.

By Proposition 4.5 and Lemma 8.2, $d_\alpha(\eta, \nu) \leq R(\mu) + G + 1$. Consider the set of curves in $\{\gamma_k\}_{k=0}^\infty$ that are contained in and fill an essential subsurface Z with the property that $\alpha \subseteq \partial Z$. Then, by $\mathcal{P}(iv)$, this set of curves is contained in a subsurface $Y \subset Z$ such that α is not a boundary component of Y .

Let $W \subset S - (Y \cup \alpha)$ be the (possibly disconnected) union of components meeting α (so two components of ∂W are isotopic to α in S). Since W contains no curves in $\{\gamma_k\}$, Proposition 4.5 and Lemma 8.2 imply that for all connected subsurfaces $V \subset W$, $d_V(\eta, \nu) \leq R(\mu) + G + 1$. By Theorem 2.7

$i(\pi_W(\eta), \pi_W(\nu))$ is bounded above (depending only on $R(\mu)$ and G). Consequently, there exists a simple closed curve ω in S intersecting α at most twice with $i(\pi_W(\omega), \pi_W(\eta))$ and $i(\pi_W(\omega), \pi_W(\nu))$ uniformly bounded (again depending on $R(\mu)$ and G). Therefore, $i(\pi_Z(\omega), \pi_Z(\eta))$ and $i(\pi_Z(\omega), \pi_Z(\nu))$ are uniformly bounded, hence so is $K(Z)$.

According to [Raf05, Theorem 6.1], there is uniform lower bound for $\text{Hyp}_t(\alpha)$. The lemma is completed by setting $\epsilon > 0$ to be any number less than this uniform lower bound. \square

In what follows, we will assume $L \geq L_0 = \exp(2W_0)\epsilon_0$ as in §9.3.

Theorem 9.15. *Let $\delta \in \mathcal{C}(S)$ be any curve and $L \geq L_0$. Then there exists $K, C, T > 0$, depending on L, δ , and $R(\mu)$ from Proposition 4.5, with the following property. Suppose $t \geq T$ and that P is an L -bounded length pants decomposition of S containing σ_k , for some k . Then for all $\alpha \in P \setminus \sigma_k$, we have*

$$i(\delta, \alpha) \stackrel{*}{\prec}_K A(0, k + m - 1).$$

and

$$\text{tw}_\alpha(\delta, X_t) \leq C.$$

Proof. We first prove the bound on intersection numbers. For any t , suppose α is part of an L -bounded length pants decomposition. Then [Raf14, Theorem 6.1] and the triangle inequality imply that for every subsurface $Z \neq Y_\alpha$, we have

$$d_Z(\eta, \alpha) + d_Z(\alpha, \nu) \stackrel{\pm}{\asymp} d_Z(\eta, \nu)$$

where the additive error depends on S and L .

We assume that $T_0 > 0$ is large enough so that for all $t \geq T_0$ there exists k so that every curve in σ_k has length at most L at time t . We write $k(t)$ for such a k . As in the proof of Lemma 8.2 and Lemma 9.13, we may take $T \geq T_0$ so that for all $t \geq T$, $d_Z(\delta, \nu) \stackrel{\pm}{\asymp} d_Z(\eta, \nu)$ for surfaces Z with $d_S(\partial Z, \gamma_{k(t)}) \leq 2$.

Now let $t \geq T$ and P be an L -bounded length pants decomposition containing $\sigma_{k(t)}$, and let Y be the component of $S \setminus \sigma_{k(t)}$ containing α and $Z \subseteq Y$ any subsurface. According to Proposition 4.5 and Lemma 8.2 we have $d_Z(\eta, \nu) \leq R(\mu) + G + 1$, and so combining the inequalities above, there exists R' (depending on $R(\mu)$ and L) so that for all surfaces $Z \subseteq Y$, we have

$$d_Z(\delta, \alpha) \leq R'.$$

Therefore, taking the threshold sufficiently large in Theorem 2.7 for the subsurface Y , there exists a constant I (depending on R' and Theorem 2.7) so that

$$i(\pi_Y(\delta), \alpha) \leq I.$$

Now, every arc of $\pi_Y(\delta)$ comes from a pair of intersection points with curves in $\sigma_{k(t)}$. Consequently, taking $\kappa(\delta)$ to be the constant from Lemma 5.11 we

have

$$i(\delta, \alpha) \leq I \sum_{d=k(t)}^{k(t)+m-1} i(\delta, \gamma_d) \stackrel{*}{\asymp}_{\kappa(\delta)} I \sum_{d=k(t)}^{k(t)+m-1} A(0, d) \leq mIA(0, k(t) + m - 1).$$

Thus, setting $K = mI\kappa(\delta)$ proves the first statement.

For the bound on twist number, we again appeal to [Raf07]—the same estimate in Theorem 9.12. Since $\alpha \notin \{\gamma_k\}_{k=0}^{\infty}$ (and α has bounded length at time $t \geq T$), we have $d_{\alpha}(\eta, \nu) \leq R(\mu) + G + 1$, where $R(\mu)$ is from Proposition 4.5 and G the constant appearing in Lemma 8.2 (from Theorem 2.11). Since the length of α is bounded below by ϵ , according to Lemma 9.14, [Raf07] implies

$$\text{tw}_{\alpha}(\delta, X_t) \leq C$$

for some $C > 0$ depending on $R(\mu), G, \epsilon$ and the surface S . \square

9.5. Second reduction and division into cases. We now consider the setup as in Theorem 9.3. As mentioned in Remark 9.4, to simplify the notation we assume $h = 0$ and $h' = 1$. It is convenient to switch to the notation $\gamma_i^h = \gamma_{im+h}$, $a_i^h = a_{im+h}$, $\sigma_i^h = \sigma_{im+h}$, etc.

We consider sequences $\{t_i\}$ with $t_i \in [\bar{a}_i^0, \bar{a}_i^1]$ for all sufficiently large i , falling into one of two possible cases:

Case 1. There exists $W > 0$ so that $t_i \in [\bar{a}_i^0, \bar{a}_i^0 + W]$.

Case 2. $\lim_{i \rightarrow \infty} t_i - \bar{a}_i^0 = \infty$.

For any curve $\delta \in \mathcal{C}(S)$ define

$$U_i^h(t, \delta) = w_t(\gamma_i^h) + \text{tw}_{\gamma_i^h}(\delta, X_t) \text{Hyp}_t(\gamma_i^h).$$

We will also fix a curve δ_0 for reference and write

$$U_i^h(t) = U_i^h(t, \delta_0).$$

The next lemma is not needed for the reduction, but for later use we make note of it now.

Lemma 9.16. *For any curve $\delta \in \mathcal{C}(S)$ and $L > 0$, there exists $T > 0$ so that for all $t \geq T$ and i, h with $\text{Hyp}_t(\gamma_i^h) \leq L$, we have*

$$U_i^h(t, \delta) \stackrel{\pm}{\asymp}_{GL} U_i^h(t).$$

Here the constant G is from Theorem 2.11 appearing in Lemma 9.13.

Proof. Given L , Lemma 9.13 provides $T > 0$ so that for all $t \geq T$, if $\text{Hyp}_t(\gamma_i^h) \leq L$, then

$$|\text{tw}_{\gamma_i^h}(\delta, X_t) - \text{tw}_{\gamma_i^h}(\delta_0, X_t)| \leq G.$$

Therefore, we have

$$|U_i^h(t, \delta) - U_i^h(t)| = |\text{tw}_{\gamma_i^h}(\delta, X_t) - \text{tw}_{\gamma_i^h}(\delta_0, X_t)| \text{Hyp}_t(\gamma_i^h) \leq GL.$$

□

We now turn to our second reduction.

Theorem 9.17. *Suppose $\{t_i\}$ is a sequence with $t_i \in [\bar{a}_i^0, \bar{a}_i^1]$ for all sufficiently large i and δ is any curve (not necessarily δ_0).*

If $\{t_i\}$ falls into Case 1, then

$$\lim_{i \rightarrow \infty} \frac{U_i^0(t_i)i(\delta, \gamma_i^0)}{\text{Hyp}_{t_i}(\delta)} = 1.$$

If $\{t_i\}$ falls into Case 2, then

$$\lim_{i \rightarrow \infty} \frac{U_i^1(t_i)i(\delta, \gamma_i^1) + U_{i+1}^0(t_i)i(\delta, \gamma_{i+1}^0)}{\text{Hyp}_{t_i}(\delta)} = 1.$$

Note in this theorem, the terms $U_j^h(t_i)$ do not depend on δ (c.f. Lemma 9.16).

Proof of Theorem 9.3 assuming Theorem 9.17. Suppose $\{t_j\}_{j=0}^\infty$ with $t_j \in [\bar{a}_{i_j}^0, \bar{a}_{i_j}^1]$ for all sufficiently large j and some i_j , so that X_{t_j} converges to some point in $\mathcal{PML}(S)$. We may pass to a subsequence so that either $t_j - \bar{a}_{i_j}^0 \leq W$ for some W , or else $t_j - \bar{a}_{i_j}^0 \rightarrow \infty$ with j . This subsequence can be viewed as a subsequence of a sequence falling into Case 1 or Case 2, respectively, and hence the conclusion of Theorem 9.17 holds for $\{t_j\}$.

Now let $\delta, \delta' \in \mathcal{C}(S)$ be any two curves. If we are in Case 1, then by Theorem 9.17 and Theorem 5.10 we have

$$\begin{aligned} \lim_{j \rightarrow \infty} \frac{\text{Hyp}_{t_j}(\delta)}{\text{Hyp}_{t_j}(\delta')} &= \lim_{j \rightarrow \infty} \frac{\text{Hyp}_{t_j}(\delta) \frac{U_{i_j}^0(t_j)i(\delta, \gamma_{i_j}^0)}{\text{Hyp}_{t_j}(\delta)}}{\text{Hyp}_{t_j}(\delta') \frac{U_{i_j}^0(t_j)i(\delta', \gamma_{i_j}^0)}{\text{Hyp}_{t_j}(\delta')}} \\ &= \lim_{j \rightarrow \infty} \frac{i(\delta, \gamma_{i_j}^0)}{i(\delta', \gamma_{i_j}^0)} = \frac{i(\delta, \bar{\nu}^0)}{i(\delta', \bar{\nu}^0)}. \end{aligned}$$

Since δ, δ' were arbitrary, it follows that $X_{t_j} \rightarrow [\bar{\nu}^0]$.

Now suppose we are in the second case. Compactness of $\mathcal{PML}(S)$ implies that by passing to a further subsequence (of the same name) the sequence

$$\{[U_{i_j}^1(t_j)\gamma_{i_j}^1 + U_{i_j+1}^0(t_j)\gamma_{i_j+1}^0]\}_{j=0}^\infty$$

converges in $\mathcal{PML}(S)$. Note that this limit is necessarily of the form

$$[y_0\bar{\nu}^0 + y_1\bar{\nu}^1] \in [[\bar{\nu}^0], [\bar{\nu}^1]]$$

by Theorem 5.10. Now observe that for all j , the numerator from Case 2 of Theorem 9.17 is given by

$$U_{i_j}^1(t_j)i(\delta, \gamma_{i_j}^1) + U_{i_j+1}^0(t_j)i(\delta, \gamma_{i_j+1}^0) = i(\delta, U_{i_j}^1(t_j)\gamma_{i_j}^1 + U_{i_j+1}^0(t_j)\gamma_{i_j+1}^0).$$

Therefore, similar to the above calculation, appealing to Theorem 9.17 we have

$$\begin{aligned} \lim_{j \rightarrow \infty} \frac{\text{Hyp}_{t_j}(\delta)}{\text{Hyp}_{t_j}(\delta')} &= \lim_{j \rightarrow \infty} \frac{i(\delta, U_{i_j}^1(t_j)\gamma_{i_j}^1 + U_{i_j+1}^0(t_j)\gamma_{i_j+1}^0)}{i(\delta', U_{i_j}^1(t_j)\gamma_{i_j}^1 + U_{i_j+1}^0(t_j)\gamma_{i_j+1}^0)} \\ &= \frac{i(\delta, y_0\bar{\nu}^0 + y_1\bar{\nu}^1)}{i(\delta', y_0\bar{\nu}^0 + y_1\bar{\nu}^1)} \end{aligned}$$

Again, because δ, δ' were arbitrary we see that X_{t_j} limits to $[y_0\bar{\nu}^0 + y_1\bar{\nu}^1]$. This completes the proof. \square

9.6. Final estimates and proof of Theorem 9.17. Here we provide the final estimates necessary for the proof of Theorem 9.17 (and hence the main theorem). The proof for each of the two cases are similar, and many of the estimates can be made simultaneously.

We assume for the remainder of the paper that $\{t_i\}$ is a sequence so that $t_i \in [\bar{a}_i^0, \bar{a}_i^1]$ for all sufficiently large i and that δ is an arbitrary curve (not necessarily our reference curve δ_0).

If we are in Case 1 with $t_i - \bar{a}_i^0 \leq W$, then by Lemma 9.8, for all sufficiently large i there exists $L \geq \exp(2W)\epsilon$ and an L -bounded length pants decomposition P_i for X_{t_i} containing σ_i^0 . Let

$$P_i^c = P_i \setminus \sigma_i^0.$$

If we are in Case 2, then by Lemma 8.4, for i sufficiently large, we have $t_i \in [a_{i+1}^0, a_{i+1}^0]$, and there exists $L \geq 0$ (depending only on S) and an L -bounded pants decomposition P_i for X_{t_i} containing σ_i^1 . Similar to Case 1, we let

$$P_i^c = P_i \setminus \sigma_i^1.$$

We use Proposition 9.6 to estimate $\text{Hyp}_{t_i}(\delta)$. Appealing to Theorem 9.15 together with Lemma 5.11 and monotonicity of $\{A(0, k)\}_{k=0}^\infty$ (Lemma 5.6) to group together all the intersection number errors in Proposition 9.6, this takes a somewhat simpler form. To write it, recall that for all $h \in \{0, \dots, m-1\}$ and $i \geq 0$, we have

$$c_i^h = A(0, im + h) = \prod_{j=0}^{i-1} be_j^h.$$

The estimates are then similar, but depend on the case:

Case 1.

$$(9.3) \quad \text{Hyp}_{t_i}(\delta) = \sum_{h=0}^{m-1} \text{Hyp}_{t_i}(\delta, \gamma_i^h) + \sum_{\alpha \in P_i^c} \text{Hyp}_{t_i}(\delta, \alpha) + O(c_i^{m-1}).$$

The O -error term depends on L (hence W) and δ , but is independent of i .

Case 2.

$$(9.4) \quad \text{Hyp}_{t_i}(\delta) = \sum_{h=1}^{m-1} \text{Hyp}_{t_i}(\delta, \gamma_i^h) + \text{Hyp}_{t_i}(\delta, \gamma_{i+1}^0) + \sum_{\alpha \in P_i^c} \text{Hyp}_{t_i}(\delta, \alpha) + O(c_{i+1}^0).$$

In this case, the O -error term depends on L (which depends only on S) and δ , but is again independent of i .

We will appeal to the various estimates previously made, specifically those in §8, §9.3, and §9.4. The first estimate involves the contributions to (9.3) and (9.4) from the curves of P_i^c .

Lemma 9.18. *For all i sufficiently large and $\alpha \in P_i^c$, we have*

$$\text{Hyp}_{t_i}(\delta, \alpha) = \begin{cases} O(c_i^{m-1}) & \text{in Case 1} \\ O(c_{i+1}^m) & \text{in Case 2.} \end{cases}$$

Proof. From (9.1) we have

$$\text{Hyp}_{t_i}(\delta, \alpha) = (w_{t_i}(\alpha) + \text{tw}_\alpha(\delta, X_{t_i}) \text{Hyp}_{t_i}(\alpha)) i(\delta, \alpha).$$

By Lemma 9.14 and Theorem 9.15, every term on the right except $i(\delta, \alpha)$ is bounded, depending on δ and L (and the resulting constants from those statements). The lemma follows. \square

Corollary 9.19. *For all i sufficiently large we have*

$$(9.5) \quad \text{Hyp}_{t_i}(\delta) = \sum_{h=0}^{m-1} \text{Hyp}_{t_i}(\delta, \gamma_i^h) + O(c_i^{m-1}) \quad \text{in Case 1}$$

$$(9.6) \quad \text{Hyp}_{t_i}(\delta) = \sum_{h=1}^{m-1} \text{Hyp}_{t_i}(\delta, \gamma_i^h) + \text{Hyp}_{t_i}(\delta, \gamma_{i+1}^0) + O(c_{i+1}^0) \quad \text{in Case 2.}$$

We write the remaining terms using the notation set in the previous section as

$$\text{Hyp}_{t_i}(\delta, \gamma_j^h) = U_j^h(t_i, \delta) i(\delta, \gamma_j^h).$$

Estimates for these terms are given in the next four lemmas.

Lemma 9.20. *For all sufficiently large i and all $1 < h \leq m-1$, we have*

$$U_i^h(t_i, \delta) \stackrel{+}{\asymp} 4 \left(\sum_{j=1}^i \log \left(\frac{e_j^0}{e_{j-1}^h} \right) + t_i - \bar{a}_i^0 \right).$$

In Case 1, this also holds for $h = 1$.

Proof. Note that for $1 < h \leq m-1$ (as well as $h = 1$ in Case 1), we have $\underline{a}_i^h < t_i < \bar{a}_i^h$, for all sufficiently large i . Therefore, $\text{Hyp}_t(\gamma_i^h) \leq \epsilon_0 < L$ and so Theorem 9.12 implies

$$\text{tw}_{\gamma_i^h}(\delta, X_{t_i}) \text{Hyp}_{t_i}(\gamma_i^h) \stackrel{*}{\prec} 1.$$

On the other hand by Lemma 9.10

$$w_{t_i}(\gamma_i^h) \stackrel{+}{\asymp} 4(t_i - \underline{a}_i^h) = 4(\bar{a}_i^0 - \underline{a}_i^h + (t_i - \bar{a}_i^0)),$$

since $\underline{a}_i^h \ll \bar{a}_i^0 \leq t_i \leq a_i^h$ (for sufficiently large i and all $1 < h \leq m-1$ in both cases, and also $h=1$ in Case 1) by Lemma 8.9. The lemma now follows from this by substituting in from (8.2), (8.3), and (8.4) and dropping constants. \square

Lemma 9.21. *Suppose $\{t_i\}$ falls into Case 1 with constant W . Then for all sufficiently large i , we have*

$$U_i^0(t_i, \delta) \stackrel{*}{\asymp} e_i^0$$

where the multiplicative error depends on W , δ , (and all resulting constants), but not i .

Proof. Because $t_i - \bar{a}_i^0 \leq W$, $\text{Hyp}_{t_i}(\gamma_i^0)$ is bounded above and below by Lemma 9.9 and Lemma 9.7, the bound depending on W . By Lemma 2.12, $w_{t_i}(\gamma_i^0)$ is also bounded. To complete the proof we note that by Theorem 9.12

$$\text{tw}_{\gamma_i^0}(\delta, t_i) \stackrel{*}{\asymp} e_i^0.$$

\square

Lemma 9.22. *Suppose $\{t_i\}$ falls into Case 2. Then for all large i , we have*

$$U_{i+1}^0(t_i, \delta) \stackrel{+}{\asymp} 4(t_i - \bar{a}_i^0).$$

Proof. This is almost identical to the proof of Lemma 9.20, so we omit it. \square

For the only remaining situation, a very coarse estimate will suffice.

Lemma 9.23. *Suppose $\{t_i\}$ falls into Case 2. Then*

$$U_i^1(t_i, \delta) \rightarrow \infty.$$

Proof. Since we are in Case 2, $t_i - \underline{a}_i^1 \geq t_i - \bar{a}_i^0 \rightarrow \infty$. Then either $t_i \leq a_i^1$ or $a_i^1 \leq t_i \leq \bar{a}_i^1$. In the former case, Lemma 9.10 shows that $w_{t_i}(\gamma_i^1) \rightarrow \infty$. In the latter case, either $w_{t_i}(\gamma_i^1) \rightarrow \infty$, and we are done, or else $w_{t_i}(\gamma_i^1)$ is bounded. If $w_{t_i}(\gamma_i^1)$ is bounded, then (2.2) implies $\text{Hyp}_{t_i}(\gamma_i^1)$ is bounded below. Since $e_i^1 \rightarrow \infty$, Theorem 9.12 implies that $\text{tw}_{\gamma_i^1}(\delta, \gamma_i^1) \rightarrow \infty$, completing the proof. \square

From these, we deduce the following

Corollary 9.24. *If $\{t_i\}$ falls into Case 1 (and hence $t_i - \bar{a}_i^0 \leq W$), then for all i sufficiently large and $1 \leq h \leq m-1$ we have*

$$(9.7) \quad \text{Hyp}_{t_i}(\delta, \gamma_i^h) \stackrel{*}{\asymp} \left(\sum_{j=1}^i \log \left(\frac{e_j^0}{e_{j-1}^h} \right) \right) \prod_{j=0}^{i-1} be_j^h, \text{ and}$$

$$(9.8) \quad \text{Hyp}_{t_i}(\delta, \gamma_i^0) \stackrel{*}{\asymp} \prod_{j=0}^i be_j^0$$

If $\{t_i\}$ falls into Case 2 (and hence $t_i - \bar{a}_i^0 \rightarrow \infty$), then for all i sufficiently large and $2 \leq h \leq m - 1$ we have

$$(9.9) \quad \text{Hyp}_{t_i}(\delta, \gamma_i^h) \stackrel{*}{\asymp} \left(\sum_{j=1}^i \log \left(\frac{e_j^0}{e_{j-1}^h} \right) + t_i - \bar{a}_i^0 \right) \prod_{j=0}^{i-1} be_j^h, \text{ and}$$

$$(9.10) \quad \text{Hyp}_{t_i}(\delta, \gamma_{i+1}^0) \stackrel{*}{\asymp} (t_i - \bar{a}_i^0) \prod_{j=0}^i be_j^0$$

The multiplicative constants depend on W (in Case 1) and δ , and all constants that depend on these.

Proof. By Lemma 5.11, there exists $\kappa(\delta) > 0$ so that

$$i(\delta, \gamma_i^h) \stackrel{*}{\asymp}_{\kappa(\delta)} A(0, im + h) = c_i^h = \prod_{j=0}^{i-1} be_j^h.$$

Since

$$\text{Hyp}_{t_i}(\delta, \gamma_j^h) = U_j^h(t_i, \delta) i(\delta, \gamma_j^h),$$

the corollary follows from Lemmas 9.20, 9.21, 9.22, and 9.23. \square

We are now ready for the

Proof of Theorem 9.17. Observe that from Lemmas 9.20, 9.21, 9.22, and 9.23, we see that for all h , as $i \rightarrow \infty$ we have

$$U_i^h(\delta, t_i) \rightarrow \infty \text{ and } U_{i+1}^0(\delta, t_i) \rightarrow \infty.$$

where the second limit is only true in Case 2, and the first is only relevant for $h = 0$ in Case 1. By Lemma 9.16, it suffices to prove Theorem 9.17 replacing all terms of the form $U_j^h(t_i)$ with terms $U_j^h(t_i, \delta)$.

The proof will use the estimates (9.5) and (9.6) from Corollary 9.19 and we divide it into the two cases.

Proof in Case 1. We look at each term on the right-hand side of (9.5) and divide by the term $\text{Hyp}_{t_i}(\delta, \gamma_i^0)$. Doing this for the terms $\text{Hyp}_{t_i}(\delta, \gamma_i^h)$ for $1 \leq h \leq m - 1$, Equations (9.7) and (9.8) imply

$$\frac{\text{Hyp}_{t_i}(\delta, \gamma_i^h)}{\text{Hyp}_{t_i}(\delta, \gamma_i^0)} \stackrel{*}{\asymp} be_0^h \left(\sum_{j=1}^i \log \left(\frac{e_j^0}{e_{j-1}^h} \right) \right) \prod_{j=1}^i \frac{e_{j-1}^h}{e_j^0} = \log \left(\prod_{j=1}^i \frac{e_j^0}{e_{j-1}^h} \right) \prod_{j=1}^i \frac{e_{j-1}^h}{e_j^0}.$$

Since $jm > (j - 1)m + h$ implies $e_j^0 \geq ae_{j-1}^h$, we have $\prod \frac{e_{j-1}^h}{e_j^0} \leq a^{-i}$, and since $a > 1$

$$\lim_{i \rightarrow \infty} \frac{\text{Hyp}_{t_i}(\delta, \gamma_i^h)}{\text{Hyp}_{t_i}(\delta, \gamma_i^0)} = 0.$$

The only remaining term, other than $\text{Hyp}_{t_i}(\delta, \gamma_i^0)$, is $O(c_i^{m-1})$. For this, we note that by definition

$$c_i^h = \prod_{j=0}^{i-1} be_j^h,$$

and therefore, for the same reason as above, we have

$$\frac{O(c_i^h)}{\text{Hyp}_{t_i}(\gamma_i^0)} \stackrel{*}{\asymp} be_0^h \prod_{j=1}^i \frac{e_{j-1}^h}{e_j^0} \rightarrow 0$$

as $i \rightarrow \infty$. Now combining all these estimates into (9.5) we have

$$\lim_{i \rightarrow \infty} \frac{\text{Hyp}_{t_i}(\delta)}{\text{Hyp}_{t_i}(\delta, \gamma_i^0)} = \lim_{i \rightarrow \infty} \sum_{h=0}^{m-1} \frac{\text{Hyp}_{t_i}(\delta, \gamma_i^h)}{\text{Hyp}_{t_i}(\delta, \gamma_i^0)} + \frac{O(c_i^{m-1})}{\text{Hyp}_{t_i}(\delta, \gamma_i^0)} = 1$$

This completes the proof since

$$\text{Hyp}_{t_i}(\delta, \gamma_i^0) = U_i^0(t_i, \delta)i(\delta, \gamma_i^0).$$

Proof in Case 2. We again look at each term on the right-hand side of (9.6) and this time begin by dividing most of the terms by $\text{Hyp}_{t_i}(\delta, \gamma_{i+1}^0)$. Doing this for the terms $\text{Hyp}_{t_i}(\delta, \gamma_i^h)$ for $2 \leq h \leq m-1$, Equations (9.9) and (9.10), together with the fact that $t_i - \bar{a}_i^0 \rightarrow \infty$, implies

$$\begin{aligned} \frac{\text{Hyp}_{t_i}(\delta, \gamma_i^h)}{\text{Hyp}_{t_i}(\delta, \gamma_{i+1}^0)} &\stackrel{*}{\asymp} \frac{be_0^h}{t_i - \bar{a}_i^0} \left(\sum_{j=1}^i \log \left(\frac{e_j^0}{e_{j-1}^h} \right) + t_i - \bar{a}_i^0 \right) \prod_{j=1}^i \frac{e_{j-1}^h}{e_j^0} \\ &\leq be_0^h \left(1 + \log \left(\prod_{j=1}^i \frac{e_j^0}{e_{j-1}^h} \right) \right) \prod_{j=1}^i \frac{e_{j-1}^h}{e_j^0}. \end{aligned}$$

Now as above, the right-hand side tends to 0 as $i \rightarrow \infty$, and hence

$$\lim_{i \rightarrow \infty} \frac{\text{Hyp}_{t_i}(\delta, \gamma_i^h)}{\text{Hyp}_{t_i}(\delta, \gamma_{i+1}^0)} = 0$$

Next we consider the $O(c_{i+1}^0)$ term of (9.6). By definition of c_{i+1}^0 , together with (9.10) and the fact that $t_i - \bar{a}_i^0 \rightarrow \infty$, as $i \rightarrow \infty$ we have

$$\frac{O(c_{i+1}^0)}{\text{Hyp}_{t_i}(\delta, \gamma_{i+1}^0)} \stackrel{*}{\asymp} \frac{\prod_{j=0}^i be_j^0}{(t_i - \bar{a}_i^0) \prod_{j=0}^i be_j^0} = \frac{1}{t_i - \bar{a}_i^0} \rightarrow 0.$$

Since $\text{Hyp}_{t_i}(\delta, \gamma_i^1) + \text{Hyp}_{t_i}(\delta, \gamma_{i+1}^0) > \text{Hyp}_{t_i}(\delta, \gamma_{i+1}^0)$, we could have divided by this larger quantity, and the above limits would still be zero. Plugging into (9.6) we deduce

$$\lim_{i \rightarrow \infty} \frac{\text{Hyp}_{t_i}(\delta)}{\text{Hyp}_{t_i}(\delta, \gamma_i^1) + \text{Hyp}_{t_i}(\delta, \gamma_{i+1}^0)} = 1.$$

Since

$$\text{Hyp}_{t_i}(\delta, \gamma_i^1) + \text{Hyp}_{t_i}(\delta, \gamma_{i+1}^0) = U_i^1(t_i, \delta)i(\delta, \gamma_i^1) + U_{i+1}^0(t_i, \delta)i(\delta, \gamma_{i+1}^0)$$

this completes the proof of Case 2, and hence of the theorem. \square

REFERENCES

- [Beh06] Jason A. Behrstock, *Asymptotic geometry of the mapping class group and Teichmüller space*, *Geom. Topol.* **10** (2006), 1523–1578.
- [BM] Jeffery Brock and Babak Modami, *Recurrent Weil-Petersson geodesics with minimal non-uniquely ergodic ending lamination*, *Geom. Topol.* to appear, arXiv:1409.1562.
- [BMM10] Jeffrey Brock, Howard Masur, and Yair Minsky, *Asymptotics of Weil-Petersson geodesics. I. Ending laminations, recurrence, and flows*, *Geom. Funct. Anal.* **19** (2010), no. 5, 1229–1257.
- [BMM11] ———, *Asymptotics of Weil-Petersson geodesics II: bounded geometry and unbounded entropy*, *Geom. Funct. Anal.* **21** (2011), no. 4, 820–850.
- [Bon88] Francis Bonahon, *The geometry of Teichmüller space via geodesic currents*, *Invent. Math.* **92** (1988), no. 1, 139–162. MR 931208 (90a:32025)
- [Bus10] Peter Buser, *Geometry and spectra of compact Riemann surfaces*, Modern Birkhäuser Classics, Birkhäuser Boston Inc., Boston, MA, 2010, Reprint of the 1992 edition.
- [CEG87] R. D. Canary, D. B. A. Epstein, and P. Green, *Notes on notes of Thurston*, Analytical and geometric aspects of hyperbolic space (Coventry/Durham, 1984), *London Math. Soc. Lecture Note Ser.*, vol. 111, Cambridge Univ. Press, Cambridge, 1987, pp. 3–92.
- [CLM12] Matt T. Clay, Christopher J. Leininger, and Johanna Mangahas, *The geometry of right-angled Artin subgroups of mapping class groups*, *Groups Geom. Dyn.* **6** (2012), no. 2, 249–278.
- [CMW] Jon Chaika, Howard Masur, and Michael Wolf, *Limits in PMF of Teichmüller geodesics*, arXiv:1406.0564.
- [CR07] Young-Eun Choi and Kasra Rafi, *Comparison between Teichmüller and Lipschitz metrics*, *J. Lond. Math. Soc.* (2) **76** (2007), no. 3, 739–756.
- [CRS08] Young-Eun Choi, Kasra Rafi, and Caroline Series, *Lines of minima and Teichmüller geodesics*, *Geom. Funct. Anal.* **18** (2008), no. 3, 698–754.
- [FLM01] Benson Farb, Alexander Lubotzky, and Yair Minsky, *Rank-1 phenomena for mapping class groups*, *Duke Math. J.* **106** (2001), no. 3, 581–597.
- [FLP79] A. Fathi, F. Laudenbach, and V. Poenaru, *Travaux de Thurston sur les surfaces*, *Astérisque* No. 66-67 (1979), 1–286.
- [Gab09] David Gabai, *Almost filling laminations and the connectivity of ending lamination space*, *Geom. Topol.* **13** (2009), no. 2, 1017–1041.
- [Gar87] Frederick P. Gardiner, *Teichmüller theory and quadratic differentials*, *Pure and Applied Mathematics* (New York), John Wiley & Sons, Inc., New York, 1987, A Wiley-Interscience Publication. MR 903027 (88m:32044)
- [GM91] F.P. Gardiner and H.A. Masur, *Extremal length geometry of Teichmüller space*, *Complex Variables Theory Appl.* **16** (1991), no. 2-3, 209–237.
- [HM79] John Hubbard and Howard Masur, *Quadratic differentials and foliations*, *Acta Math.* **142** (1979), no. 3-4, 221–274.
- [Iva92] Nikolai V. Ivanov, *Subgroups of Teichmüller modular groups*, *Translations of Mathematical Monographs*, vol. 115, American Mathematical Society, Providence, RI, 1992, Translated from the Russian by E. J. F. Primrose and revised by the author.
- [Kla] Erica Klarreich, *The boundary at infinity of the curve complex*, Preprint, 1999.
- [Len08] Anna Lenzen, *Teichmüller geodesics that do not have a limit in PMF*, *Geom. Topol.* **12** (2008), no. 1, 177–197. MR 2377248 (2008k:30056)

- [Lev83] Gilbert Levitt, *Foliations and laminations on hyperbolic surfaces*, *Topology* **22** (1983), no. 2, 119–135.
- [LLR] Christopher Leininger, Anna Lenzhen, and Kasra Rafi, *Limit sets of Teichmüller geodesics with minimal non-uniquely ergodic vertical foliation*, *J. Reine Angew. Math.* to appear, arXiv: 1312.2305.
- [LM10] Anna Lenzhen and Howard Masur, *Criteria for the divergence of pairs of Teichmüller geodesics*, *Geom. Dedicata* **144** (2010), 191–210.
- [Man13] Johanna Mangahas, *A recipe for short-word pseudo-Anosovs*, *Amer. J. Math.* **135** (2013), no. 4, 1087–1116. MR 3086070
- [Mas82] Howard Masur, *Two boundaries of Teichmüller space*, *Duke Math. J.* **49** (1982), no. 1, 183–190.
- [Mas85] Bernard Maskit, *Comparison of hyperbolic and extremal lengths*, *Ann. Acad. Sci. Fenn. Ser. A I Math.* **10** (1985), 381–386. MR 802500 (87c:30062)
- [Mas92] Howard Masur, *Hausdorff dimension of the set of nonergodic foliations of a quadratic differential*, *Duke Math. J.* **66** (1992), no. 3, 387–442.
- [Min96] Yair N. Minsky, *Extremal length estimates and product regions in Teichmüller space*, *Duke Math. J.* **83** (1996), no. 2, 249–286. MR 1390649 (97b:32019)
- [MM99] Howard A. Masur and Yair N. Minsky, *Geometry of the complex of curves. I. Hyperbolicity*, *Invent. Math.* **138** (1999), no. 1, 103–149.
- [MM00] H. A. Masur and Y. N. Minsky, *Geometry of the complex of curves. II. Hierarchical structure*, *Geom. Funct. Anal.* **10** (2000), no. 4, 902–974.
- [Mod15] Babak Modami, *Prescribing the behavior of Weil–Petersson geodesics in the moduli space of Riemann surfaces*, *J. Topol. Anal.* **7** (2015), no. 4, 543–676.
- [PH92] R. C. Penner and J. L. Harer, *Combinatorics of train tracks*, *Annals of Mathematics Studies*, vol. 125, Princeton University Press, Princeton, NJ, 1992.
- [Raf05] Kasra Rafi, *A characterization of short curves of a Teichmüller geodesic*, *Geom. Topol.* **9** (2005), 179–202.
- [Raf07] ———, *A combinatorial model for the Teichmüller metric*, *Geom. Funct. Anal.* **17** (2007), no. 3, 936–959.
- [Raf14] ———, *Hyperbolicity in Teichmüller space*, *Geom. Topol.* **18** (2014), no. 5, 3025–3053.
- [Thu86] W.P. Thurston, *Geometry and topology of 3-manifolds*, Princeton University Lecture Notes, online at <http://www.msri.org/publications/books/gt3m>, 1986.
- [Wol79] Scott Wolpert, *The length spectra as moduli for compact Riemann surfaces*, *Ann. of Math. (2)* **109** (1979), no. 2, 323–351. MR 528966 (80j:58067)

DEPARTMENT OF MATHEMATICS, BROWN UNIVERSITY, PROVIDENCE, RI,
E-mail address: brock@math.brown.edu

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ILLINOIS AT URBANA-CHAMPAIGN,
 1409 W GREEN ST, URBANA, IL
E-mail address: clein@illinois.edu

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ILLINOIS AT URBANA-CHAMPAIGN,
 1409 W GREEN ST, URBANA, IL
E-mail address: bmodami@illinois.edu

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF TORONTO, TORONTO, ON
E-mail address: rafi@math.utoronto.edu