Algebraic limits of geometrically finite manifolds are tame

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1 Introduction

An irreducible 3-manifold M is *tame* if it is homeomorphic to the interior of a compact 3-manifold. In this paper we address the following conjecture of A. Marden.

Marden's Tameness Conjecture. *Let M be a complete hyperbolic 3-manifold with finitely generated fundamental group. Then M is tame.*

In his 1986 article [Th2], W. Thurston proposed that one might approach Marden's conjecture from a dynamical point of view via a study of limits in the natural deformation space: its interior consists of geometrically finite manifolds, and promoting their tameness to *algebraic limits* on the boundary has proven to be a successful strategy to address Marden's conjecture in special cases. In this paper we complete this part of his approach.

Theorem 1.1. *Each algebraic limit of geometrically finite hyperbolic 3-manifolds is tame.*

Theorem 1.1 reduces Marden's tameness conjecture to the density conjecture of Bers, Sullivan, and Thurston, which predicts that every hyperbolic 3-manifold with finitely generated fundamental group is an algebraic limit of geometrically finite hyperbolic 3-manifolds.

Our result finishes the cases left unaddressed by our previous work with K. Bromberg and R. Evans (see [BBES]). We first outline some further consequences of our results and then review the history of our approach.

Strong convergence. A central difficulty arising in the consideration of algebraic convergence is the lack of continuity of many important geometric and topological properties. A better topology for understanding geometric changes under deformations is the Gromov-Hausdorff or

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geometric topology on the set $\{(M,\omega)\}$ of complete hyperbolic 3-manifolds M equipped with *base-frames* ω, specified by a choice of orthonormal frame at a basepoint.

After passing to a subsequence, manifolds in an algebraically convergent sequence $M_n \to M$ may be equipped with base-frames ω_n so that the sequence (M_n, ω_n) converges geometrically to a limit (N_G, ω_G) with a locally isometric cover

$$\pi\colon (M,\omega) \to (N_G, \omega_G)$$

by the natural framed algebraic limit (M,ω). When π is an isometry and the algebraic and geometric limits agree, we say the convergence to the limit is *strong*.

If cusps of the limit M of a sequence M_n correspond algebraically to cusps of the approximates, we say the convergence $M_n \to M$ is *type-preserving*. This condition is motivated by a conjecture of Jørgensen.

Jørgensen's Conjecture. *Let (M_n) be an algebraically convergent sequence of geometrically finite hyperbolic 3-manifolds with limit M. If the convergence $M_n \to M$ is type-preserving then it is strong.*

In [BBES], our strategy was to show that each algebraic limit of geometrically finite manifolds is approximated by a type-preserving sequence $M_n \to M$. Applying cases of Jørgensen's conjecture due to Anderson and Canary [AC1, AC2], tameness of the limit M follows provided M has non-empty conformal boundary by results of Canary-Minsky and Evans [CM, Ev]. Here, while we employ a similar philosophy of improving our approximating sequences, we engage in a more specific topological investigation of the geometric limit to show tameness of the limit M directly. As a corollary we confirm Jørgensen's conjecture.

Theorem 1.2. *Let (M_n) be a sequence of geometrically finite manifolds converging in a type-preserving manner to a limit M. Then the convergence of (M_n) to M is strong.*

Proof. When M has non-empty conformal boundary, the theorem follows from the main results of [AC1, AC2]. Otherwise, applying Theorem 1.1, the limit M is tame, and the strong convergence follows from [Can3, Theorem 9.2] □

Conformal dynamical systems. Theorem 1.1 has important dynamical consequences for the action of the associated Kleinian group on $\widehat{\mathbb{C}}$. In [BBES] we applied the work of Thurston, Bonahon, and Canary (see [Th1, Bon, Can2]) to show that for each algebraic limit $M = \mathbb{H}^3/\Gamma$ of geometrically finite hyperbolic 3-manifolds the Ahlfors measure conjecture holds, namely that either the limit set $\Lambda(\Gamma)$ is all of $\widehat{\mathbb{C}}$ or $\Lambda(\Gamma)$ has Lebesgue measure zero on $\widehat{\mathbb{C}}$.

The tameness of general algebraic limits gives stronger consequences.

Corollary 1.3. *Let $M = \mathbb{H}^3/\Gamma$ be an algebraic limit of geometrically finite manifolds. Then either $\Lambda(\Gamma)$ has measure zero, or $\Lambda(\Gamma) = \widehat{\mathbb{C}}$ and Γ acts ergodically on $\widehat{\mathbb{C}} \times \widehat{\mathbb{C}}$.* □

The latter conclusion guarantees the geodesic flow on the unit tangent bundle $T_1(M)$ is ergodic, as well as the non-existence of measurable Γ-invariant line fields on $\Lambda(\Gamma)$ established in more general contexts by Sullivan early on (cf. [Sul, Th1, Mc]).

Deformation spaces. In the course of the proof of Theorem 1.1 we verify certain conjectural features of the deformation space of hyperbolic 3-manifolds with a given homotopy type. In particular, Thurston conjectured that each algebraic limit of geometrically finite manifolds is a strong limit of a perhaps different sequence of such manifolds (see [Th3]). After [BBES, Theorem 1.9], each limit of geometrically finite manifolds is a type-preserving limit of geometrically finite manifolds, so Thurston's conjecture then follows from Theorem 1.2.

Theorem 1.4. *Let M be an algebraic limit of geometrically finite hyperbolic 3-manifolds. Then there is a strongly convergent sequence $M_n \to M$ with M_n geometrically finite.* □

Also of interest in the proof of the main theorem are certain uniform estimates we obtain for limits of surfaces in 3-manifolds. We plan in a future paper to employ these estimates to study the relationship between the geometry of closed hyperbolic 3-manifolds and the combinatorics of their Heegaard splittings.

History. Our main theorem is a culmination of a series of similar results. In his original lecture notes [Th1], Thurston introduced the notion of *geometric* tameness for an incompressible end of a hyperbolic 3-manifold M. This condition posits the existence of closed geodesics exiting the end that are homotopic to simple curves on the boundary of a compact core for M.

Thurston showed that this notion of geometric tameness persists in limits of geometrically finite hyperbolic 3-manifolds with freely-indecomposable fundamental group and with certain assumptions on cusps. He showed moreover that the *topological* tameness predicted by Marden's conjecture follows in these cases, as well as the dynamical conclusions of Corollary 1.3. F. Bonahon later proved geometric tameness holds generally for hyperbolic 3-manifolds with incompressible ends [Bon], but the compressible case has remained open.

In [Can2], R. Canary demonstrated that topological tameness always guarantees geometric tameness for hyperbolic 3-manifolds, once this notion is appropriately generalized to the setting of compressible ends, and also that the conclusions of Corollary 1.3 hold as a consequence. The condition of topological tameness has been a central focus since this work. Renewing the limiting approach, Canary and Y. Minsky [CM] established that tameness persists in cusp-free limits of cusp-free hyperbolic manifolds, under the extra assumption that the convergence is strong. Work of R. Evans [Ev] generalized these results to the type-preserving (and *weakly* type-preserving) setting.

Recent developments in the deformation theory of hyperbolic cone-manifolds have improved our ability to choose a desirable sequence of approximating manifolds for a given limit of geometrically finite manifolds. Indeed, this is the central technique of our recent work with K. Bromberg and R. Evans in [BBES], which applies a *drilling theorem* of [BB] to establish that each algebraic limit of geometrically finite manifolds has a sequence of type-preserving approximates.

Combining this fact with theorems of Anderson and Canary (see [AC1, AC2]) giving criteria for algebraic and geometric limits to agree (cf. Theorem 1.2), the main theorem of [Ev] guarantees that the limit M is tame whenever either

1. M has non-empty conformal boundary, or
2. $\pi_1(M)$ is not a compression body group

(recall G is a *compression body group* if it is isomorphic to a non-trivial free product of orientable surface groups and infinite cyclic groups). It is the remaining recalcitrant case that the limit M has empty conformal boundary and $\pi_1(M)$ is a compression body group that we address in the present treatment.

Plan of the paper. We describe the plan of the paper and suggest the structure of the argument. In section 2 we give a condition on a sequence that guarantees that all cusps of the geometric limit of a sequence M_n correspond *geometrically* to cusps in M_n; like the type-preserving condition for algebraic convergence, this condition gives us substantially more control over degenerations that can occur. Section 3 proves a combination theorem for tame hyperbolic manifolds along essential incompressible surfaces.

Applying these techniques, in section 4 we give the proof of the main theorem assuming the existence of a tame degenerate (relative) end E in the geometric limit bounded by a surface S that is either compressible or incompressible with an essential curve that is homotopic to a cusp by a homotopy that intersects the core essentially. Cutting along essential disks or annuli we decompose $\pi_1(M)$ into subgroups with non-empty domain of discontinuity, reducing the theorem to the main theorem of [BBES].

The remainder of the paper is devoted to finding the surface S and the tame end E of the geometric limit bounded by S. The techniques here are generalizations of the interpolation arguments of Canary-Minsky, together with a crucial application of the geometrically type-preserving condition defined in section 2 to make the arguments work in the presence of cusps.

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Remark: During this manuscript's final stages of preparation, Ian Agol announced a proof of the tameness conjecture, and Danny Calegari and David Gabai announced an indepenent proof shortly thereafter.

2 Deformations, drilling and strong convergence

In this section we establish the central tool that will allow us to control the behavior of parabolics in an algebraically convergent sequence. We begin by introducing the necessary background on algebraic and geometric convergence, and then discuss how to find a sequence approximating a given algebraic limit M of geometrically finite manifolds in such a way that the cusps of the *geometric* limit correspond to cusps in the approximates M_n.

Background. Recall, a *Kleinian group* is a discrete torsion-free subgroup of $\text{Isom}^+\mathbb{H}^3$, the orientation-preserving isometries of hyperbolic 3-space. To fix notation, we will refer to the quotient $M = \mathbb{H}^3/\Gamma$ of hyperbolic 3-space by a Kleinian group as a *hyperbolic 3-manifold*, assuming implicitly that M is complete and oriented by the standard orientation on $\mathbb{H}^3$. Unless otherwise stated, all Kleinian groups will be assumed non-elementary. The extension of the action of Γ to the Riemann sphere partitions $\widehat{\mathbb{C}}$ into its *domain of discontinuity* Ω where Γ acts properly discontinuously and its complementary limit set $\Lambda = \widehat{\mathbb{C}} \setminus \Omega$. We denote by ∂M the *conformal boundary* Ω/Γ obtained as the quotient of Ω.

Deformation spaces and the geometric topology. Let N be a compact 3-manifold whose interior $\text{int}(N)$ is homeomorphic to $\mathbb{H}^3/\Gamma$ for some Kleinian group Γ. Then the representation space

$$AH(N) = \{\rho\colon \pi_1(N) \to \text{Isom}^+\mathbb{H}^3 \mid \rho \text{ is discrete and 1-1 }\}/\text{conj}.$$

parameterizes complete hyperbolic 3-manifolds homotopy equivalent to N. Convergence $\rho_n \to \rho$ of such representations is called *algebraic convergence*, and $AH(N)$ inherits the *algebraic topology* as the quotient of the topology of algebraic convergence. The convergence $\rho_n \to \rho$ is called *type-preserving* when $\rho_n(g)$ is parabolic if and only if $\rho(g)$ is as well.

As each conjugacy class in $AH(N)$ determines a complete hyperbolic 3-manifold up to isometry, we will often refer to the hyperbolic manifold itself as an element $M \in AH(N)$ assuming an implicit isomorphism $\rho\colon \pi_1(N) \to \pi_1(M)$.

Unfortunately, fine geometric information can be lost in the passage to limits. Each algebraically convergent sequence $\rho_n \to \rho$ admits a subsequence that converges *geometrically* as well: if $\rho_n(\pi_1(N)) = \Gamma_n$, then (Γ_n) converges geometrically to its Kleinian *geometric limit* Γ_G if

1. for each $\gamma \in \Gamma_G$ we have $\gamma_n \to \gamma$ for some $\gamma_n \in \Gamma_n$, and
2. if a subsequence (γ_{n_j}) converges then its limit lies in Γ_G.

It is evident that the limit representation $\rho(\pi_1(N))$ is a subgroup of Γ_G when Γ_n converges geometrically to Γ_G.

A complete hyperbolic 3-manifold M determines a Kleinian group only up to conjugacy; the additional data of a *base-frame* ω, namely an orthonormal frame at a basepoint determines a unique Kleinian group via the condition that the standard base-frame $\widetilde{\omega} \in \mathbb{H}^3$ descends to ω under the locally isometric covering projection $(\mathbb{H}^3, \widetilde{\omega}) \to (M, \omega)$. Then a sequence of such *framed* hyperbolic 3-manifolds (M_n, ω_n) converges geometrically to its *geometric limit* $(N_G, \omega) = (\mathbb{H}^3, \widetilde{\omega})/\Gamma_G$ if the associated Kleinian groups converge geometrically to Γ_G.

A more geometric formulation of geometric convergence of a sequence (M_n, ω_n) of framed hyperbolic manifolds to the limit (N_G, ω_G) is the existence of a sequence $\phi_n\colon K_n \to M_n$ of smooth embeddings defined on an exhaustion of N_G by compact subsets K_n with $\omega_G \in K_n$ so that for each i and $n \geq i$ the mappings ϕ_n have 1-jet sending ω_G to ω_n and ϕ_n converge C_∞ on K_i to an isometry. We call these associated mappings *virtually defined almost isometries* and use the notation

$$\phi_n : N_G \dashrightarrow M_n$$

to refer to these mappings and their implicitly defined compact domains $K(\phi_n) = K_n$.

The sequence $(M_n) \subset AH(N)$ converges *strongly* to a limit $M \in AH(N)$ if M_n converges to M in $AH(N)$ and there are base-frames $\omega_n \in M_n$ and $\omega \in M$ so that (M_n, ω_n) converge geometrically to (M, ω). As a point of terminology, we will say that an algebraically convergent sequence $M_n \to M$ *converges geometrically to a limit N_G covered by M* to refer to the existence of base-frames for which (M_n, ω_n) converges geometrically to (N_G, ω_G), and N_G is locally isometrically covered by M.

The thick-thin decomposition. By the Margulis lemma (see [BP, Thm. D.3.3]), there is a uniform constant $\mu > 0$, so that for any $\varepsilon < \mu$ and any complete hyperbolic 3-manifold M, each component T of the ε-thin part $M^{\leq \varepsilon}$ of M where the injectivity radius is at most ε has a standard form: either

1. T is a *Margulis tube*: a solid torus neighborhood $\mathbb{T}^{\varepsilon}(\gamma)$ of a short geodesic γ in M with $\ell_M(\gamma) < 2\varepsilon$ (T is the short geodesic itself if $\ell_M(\gamma) = 2\varepsilon$), or

2. T is a *cusp*: the quotient of a horoball $B \subset \mathbb{H}^3$ by the action of a $\mathbb{Z}$ or $\mathbb{Z} \oplus \mathbb{Z}$ parabolic subgroup of $\mathrm{Isom}^+ \mathbb{H}^3$ with fixed point at $\overline{B} \cap \widehat{\mathbb{C}}$.

When $T = B/\mathbb{Z} \oplus \mathbb{Z}$, the component T is called a *rank-2 cusp*, and when $T = B/\mathbb{Z}$, T is called a *rank-1 cusp*. The constant μ is called the *3-dimensional Margulis constant*.

Given a complete hyperbolic manifold M and $\varepsilon < \mu$, we will typically denote by P^{ε} the *cuspidal ε-thin part of* M, namely, the union of components of $M^{\leq \varepsilon}$ corresponding to cusps of M, and we will frequently denote single cusp components of $M^{\leq \varepsilon}$ by $\mathbb{P}^{\varepsilon}$. When reference to the hyperbolic manifold is required we will use the notation $P^{\varepsilon}(M)$, though we will often omit the reference to M when there is no danger of confusion.

Given a complete hyperbolic 3-manifold M with cusps P, we will refer to the complement $M \setminus \mathrm{int}(P)$ as the *pared submanifold* of M. Each end of the pared submanifold of M will be called a *relative end* of M. A relative end of M is *degenerate* if it has a neighborhood that embeds in the convex core of M, and it is *tame* if it has a neighborhood homeomorphic to $S \times \mathbb{R}_+$ that is a component of the complement of a compact surface $(S, \partial S) \hookrightarrow (M, \partial P)$. When such a surface determines a neighborhood of a relative end of M, we will frequently make the usual notational abuse that refers to such a neighborhood of an end as the "end" itself.

The hyperbolic 3-manifold M is *geometrically finite* if its convex core $CC(M)$, namely, the minimal geodesically convex subset of M whose inclusion is a homotopy equivalence, has finite volume. The *convex core boundary* $\partial CC(M)$ is a collection of finite-area hyperbolic Riemann surfaces with the intrinsic path metric induced from M. By a theorem of Marden [Mar], geometrically finite hyperbolic 3-manifolds are tame; for geometrically finite M the interior $\text{int}(CC(M))$ is homeomorphic to M (see [Th1, EM]).

When the geometrically finite manifold M has cusps, the convex core $CC(M)$ is naturally compactified by adjoining compact annuli and tori at infinity corresponding to each cusp of M (otherwise M is *convex cocompact*). We denote by $\mathscr{M}$ this compactification, and by the pair $(\mathscr{M}, \mathscr{P})$ the *associated pared manifold* for M, where $\mathscr{P} \subset \partial \mathscr{M}$ denotes the union of such parabolic annuli and tori in $\partial \mathscr{M}$, the *parabolic locus*. The pared manifold $(\mathscr{M}, \mathscr{P})$ plays primarily the role of recording topological and algebraic information concerning cusps of the manifold M.

Embedding cores in the geometric limit. A *compact core* of a 3-manifold N is a compact submanifold C such that the inclusion $C \hookrightarrow N$ is a homotopy equivalence. If the manifold N has boundary ∂N then a *relative compact core* is a compact core C such the inclusion of $C \cap \partial N$ in ∂N is also a homotopy equivalence. By a theorem of Peter Scott [Sco], each irreducible 3-manifold with finitely generated fundamental group admits a compact core.

After [BBES], we may constrain our investigation to algebraic limits M homotopy-equivalent to a *compression body*, namely, a compact, irreducible, orientable 3-manifold N that has a privileged boundary component $\partial_{\text{ext}} N$ called the *exterior boundary* such that $\pi_1(\partial_{\text{ext}} N)$ surjects onto $\pi_1(N)$. The remaining components $\partial N \setminus \partial_{\text{ext}} N$ are called the *interior boundary* of N and denoted $\partial_{\text{int}} N$. Each component of $\partial_{\text{int}} N$ is incompressible.

Remark: In fact, we may further constrain our working assumptions, but we must first pause to address an omitted case of [BBES]. Though the hypotheses of [BBES] focus on algebraic limits M that are not homotopy equivalent to a compression body, its techniques are sufficient to cover all cases in which the algebraic limit M has a compact core that is not *homeomorphic* to a compression body. To see this, note that in this context the remark following Corollary 3.3 of [AC2] applies to show that any type-preserving sequence $M_n \to M$ converges strongly, and thus M is tame provided each M_n is geometrically finite by the main theorem of [Ev]. Since Theorem 1.9 of [BBES] guarantees the existence of a type-preserving sequence converging to M, it follows that M is tame provided its compact core is not homeomorphic to a compression body.

Following the above remark, many of our preparatory discussions will be constrained to treat the case when the limit M has a compact core homeomorphic to a compression body. In particular, we now prove the following Lemma allowing us to choose in certain cases a core for such an M that embeds in the geometric limit (cf. [AC1, AC2]). We remark that the proof of this lemma was suggested to us by Richard Evans.

Lemma 2.1. *Let $M_n \to M$ be an algebraically convergent sequence with geometric limit N_G covered by M, and assume M has empty conformal boundary. If M has a compact core C_0 homeomorphic to a compression body, then there is a compact core C for M that embeds in N_G under the covering projection $\pi \colon M \to N_G$.*

Proof. Let

$$\partial_{\text{int}} C_0 = \partial_1 C_0 \sqcup \ldots \sqcup \partial_k C_0$$

be the interior boundary of C_0. The topology of the ends cut off by $\partial_1 C_0, \ldots, \partial_k C_0$ is understood after work of Bonahon (see [Bon]), which we record for future reference.

Bonahon's Tameness Theorem. *Let M be a complete hyperbolic 3-manifold with cuspidal thin part P. If each end of the pared submanifold $M \setminus \text{int}(P)$ is incompressible then M is tame.*

It follows that the boundary components $\partial_1 C_0, \ldots, \partial_k C_0$ of $\partial_{\text{int}} C_0$ bound ends $E_1, \ldots, E_k$ of M with $E_i \simeq \partial_i C_0 \times \mathbb{R}_+$. Further, all of these ends are degenerate by our assumption that the conformal boundary of M is empty. We apply the covering theorem of Thurston and Canary (see [Th1, Can3]).

The Covering Theorem. *Let M be a complete hyperbolic 3-manifold with parabolic locus P and let N be a hyperbolic 3-manifold covered by M by a local isometry $\pi : M \to N$. Then if E is a tame degenerate end of $M \setminus P$ then either N has finite volume and fibers over the circle, the restriction $\pi|_E$ is finite-to-one.*

The covering theorem together with an application of [JM, Lemma 3.6] (see, e.g. [KT, AC1]) implies that the locally isometric cover from an algebraic limit to a corresponding geometric limit is an embedding on each tame degenerate relative end. Hence, there is an $a \in \mathbb{R}_+$ such that the subset $\partial_i C_0 \times [a, \infty)$ of E_i embeds under π for $i = 1, \ldots, k$. After increasing a we may assume that $\pi(\partial_i C_0 \times [a, \infty)) \cap \pi(\partial_j C_0 \times [a, \infty)) = \emptyset$ for all $i \neq j$. Up to changing C_0 by an isotopy we may assume that $a = 0$, so E_i itself embeds.

Set $K = \cup_{i=1}^k \partial_i C_0 \times [0, 1]$. Now we can choose a graph $G \subset M \setminus \cup_{i=1}^k E_i$ which intersects K only in its endpoints, with $K \cup G$ connected and such that the induced homomorphism $\pi_1(K \cup G) \to \pi_1(M)$ is an isomorphism. Further, a general position argument shows that we can isotope G to guarantee that π is an embedding when restricted to G.

Assume that $\pi(G)$ intersects $\pi(E_i)$ for some $i = 1, \ldots, k$. Since $\pi(E_i)$ is homeomorphic to $\partial_i C \times \mathbb{R}_+$ we can isotope $\pi(G)$ relative to its endpoints in N_G to a graph which only intersects $\cup_i \pi(E_i)$ at its endpoints. This isotopy lifts to an isotopy from G to a graph G' such that $K \cup G'$ is connected and embeds under π and such that $\pi_1(K \cup G') \to \pi_1(M)$ is an isomorphism. Any regular neighborhood C of $K \cup G'$ is a compact core of M and if it is small enough then it embeds under π. $\square$

Recall that we have the virtually defined maps

$$\phi_n : N_G \dashrightarrow M_n.$$

When M is as in the above Lemma, the submanifold $C_n = \phi_n(C) \subset M_n$ is a compact core for all sufficiently large n, say for all n.

Uniform length decay. We now define a condition on a geometrically convergent sequence that will give us a substantially greater degree of control on the degenerations that can occur in the geometric limit.

Definition 2.2. *A sequence (M_n, ω_n) of framed hyperbolic 3-manifolds has* uniform length decay *if for every n and each $R > 0$ there is an $\varepsilon > 0$, so that if the R-ball $B_R(\omega_n) \subset M_n$ intersects a Margulis tube $\mathbb{T}^{\mu}_{\alpha}$ about a closed geodesic α, then we have*

$$\ell(\alpha) > \varepsilon.$$

In a similar spirit to the argument of [BBES], we employ the drilling theorem of [BB] to prove that each limit of geometrically finite manifolds is approximated by a sequence with uniform length decay.

Theorem 2.3. *Let $M_n \in AH(N)$ be an algebraically convergent sequence of geometrically finite manifolds with algebraic limit M. Then there is a sequence $(\hat{M}_n)$ converging algebraically to M and base-frames $\omega_n \in \hat{M}_n$ so that $(\hat{M}_n, \omega_n)$ converges geometrically to a geometric limit covered by M and $(\hat{M}_n, \omega_n)$ has uniform length decay.*

The condition of uniform length decay is readily seen to be equivalent to the following condition.

Definition 2.4. *Let (M_n, ω_n) be a geometrically convergent sequence with limit (N_G, ω_G). The convergence $(M_n, \omega_n) \to (N_G, \omega_G)$ is* geometrically type-preserving *if for each cusp $\mathbb{P} \subset N_G$, there is a horocyclic loop $\gamma \subset \mathbb{P}$ of length $\mu/2$ so that if $\phi_n \colon N_G \dashrightarrow M_n$ are the corresponding virtually defined almost isometries then $\phi_n(\gamma)$ lies in a cusp of M_n for all n sufficiently large.*

We note that if M_n lie in $AH(N)$ and the convergence $(M_n, \omega_n) \to (N_G, \omega_G)$ is geometrically type-preserving then cusps in the algebraic limit correspond (algebraically) to cusps in the approximates; such a sequence is said to be *weakly* type-preserving. Observe that the number of cusps in the geometric limit is bounded in terms of the topology of ∂N.

The final theorem this section shows that any algebraically convergent sequence $M_n \to M$ of geometrically finite hyperbolic 3-manifolds can be replaced by a geometrically type-preserving sequence with the same algebraic limit. Our method of proof is very similar to the proof of [BBES, Theorem 1.9] as it employs repeated applications of the following version of the drilling theorem of [BB, Theorem 1.3].

The Drilling Theorem. *Given $L > 1$ and $\varepsilon_0 < \mu$, there is an $\ell > 0$ so that if M is a geometrically finite hyperbolic 3-manifold and η is a closed geodesic in M with length at most ℓ, then there is an L-bi-Lipschitz diffeomorphism of pairs*

$$h\colon \left(M \setminus \mathbb{T}^{\varepsilon_0}_{\eta}, \partial \mathbb{T}^{\varepsilon_0}_{\eta}\right) \to \left(M^0 \setminus \mathbb{P}^{\varepsilon_0}_{\eta}, \partial \mathbb{P}^{\varepsilon_0}_{\eta}\right)$$

where M^0 is the complete hyperbolic structure on $M \setminus \eta$ with the same conformal boundary, and $\mathbb{P}^{\varepsilon_0}_{\eta}$ is the rank-2 cusp component of the thin part $(M^0)^{\leq \varepsilon_0}$ corresponding to η.

Remark: We remark that in [BB], we assume M has no rank-1 cusps, but in our setting this assumption can be avoided by an application of Theorem 3.4 of [BBES] together with a diagonal argument.

Proof of Theorem 2.3. Since a type-preserving strongly convergent sequence has uniform length decay, it suffices by results of [BBES] and the remark preceding Lemma 2.1 to consider the case that M has empty conformal boundary and a compact core homeomorphic to a compression body. We choose base-frames $\omega_n \in M_n$ so that (M_n, ω_n) converges geometrically to a geometric limit (N_G, ω_G). Given $R > 0$, let $B_R(\omega_G)$ denote the ball of radius R about ω_G in N_G. Then we define a new sequence M_n^R of geometrically finite manifolds in $AH(N)$ converging algebraically to M with base-frames $\omega_n^R \in M_n^R$ so that the based manifolds (M_n^R, ω_n^R) converge geometrically to a geometric limit (N_G^R, ω_G^R) covering (N_G, ω_G) by a local isometry.

First, we let $\phi_n \colon (N_G, \omega_G) \dashrightarrow (M_n, \omega_n)$ be the virtually defined almost isometries coming from the geometric convergence of (M_n, ω_n) to (N_G, ω_G), and let n_R be chosen so that the ball $B_R(\omega_G)$ lies in the domain $K(\phi_n)$ of ϕ_n for all $n > n_R$.

The sequence M_n^R is defined as follows. Let P_G be the cuspidal thin part of the geometric limit N_G, and let P_G^R be the union of components of P_G intersecting $B_R(\omega_G)$. Let $\mathbb{P}_1, \ldots, \mathbb{P}_k$ be the connected components of P_G^R. Given $\ell < \mu$, for each $\mathbb{P}_j$ let γ_j denote a closed horocyclic loop of length ℓ in $\mathbb{P}_j$. Then γ_j determines a parabolic annulus $A_j \cong S^1 \times \mathbb{R}^+$ foliated by horocyclic loops so that A_j is the totally geodesic embedding of a 2-dimensional hyperbolic cusp. For $\delta \leq \ell$, let $A_j(\delta) \subset A_j$ denote the compact subset bounded by γ_j and by the unique horocyclic loop in A_j of length δ.

Choose $R(\delta) \geq R$ so that $A_j(\delta)$ lies in $B_{R(\delta)}(\omega_G)$ for each j. Choosing ℓ sufficiently small and taking any $\delta \leq \ell$, for all $n > n_{R(\delta)}$ sufficiently large we have that $\phi_n(A_j(\delta))$ lies entirely within a component of $(M_n)^{<\mu}$ and represents an essential element of $\pi_1(M_n)$.

For such a choice of ℓ, let $\gamma_j(n)$ be the homotopy class determined by $\phi_n(\gamma_j)$. It follows that the length

$$\ell_{M_n}(\gamma_j(n)) \to 0$$

as n tends to ∞.

We may apply the drilling theorem to drill the curves $\gamma_j(n)$ out of M_n. For reference, let $\mathscr{C}_n = \gamma_1(n)^* \sqcup \ldots \sqcup \gamma_j(n)^*$, and let $\mathbb{T}_n^\varepsilon$ denote the union of ε-Margulis tubes about the curves in $\mathscr{C}_n$. Given any $\varepsilon < \mu$, there are bi-Lipschitz constants $L_n \to 1^+$, so that for n sufficiently large we have L_n-bi-Lipschitz diffeomorphisms

$$\Phi_n^{R,\varepsilon} \colon M_n \setminus \mathbb{T}_n^\varepsilon \to M_n^0 \setminus \mathbb{P}_n^\varepsilon$$

where M_n^0 denotes the complete structure on $M_n \setminus \mathscr{C}_n$, and $\mathbb{P}_n^\varepsilon$ is the union of cusps corresponding to $\mathscr{C}_n$.

Choosing ε sufficiently small so that the ball $B_{R(\delta)}(\omega_G)$ lies in $N_G^{\geq 2\varepsilon}$, for all n sufficiently large the composition

$$\Phi_n^{R,\varepsilon} \circ \phi_n \colon B_{R(\delta)}(\omega_G) \to M_n^0$$

gives well defined maps of the $R(\delta)$-ball about ω_G into the drilling M_n^0 that tend to an isometry. Letting $\omega_n^R = \Phi_n^{R,\varepsilon} \circ \phi_n(\omega_G)$, we have that the manifolds (M_n^0, ω_n^R) converge geometrically to the original geometric limit (N_G, ω_G).

By Lemma 2.1 there is a compact core C of M that embeds in N_G. Then the mappings above give bi-Lipschitz embeddings of C into M_n^0 with bi-Lipschitz constant tending to 1. Let C_n denote the images of these cores. Then the covers of M_n^0 corresponding to C_n converge algebraically to M. We denote these covers by M_n^R and denote again by ω_n^R the lifted base-frames. Then geometric limit (N_G^R, ω_G^R) of (M_n^R, ω_n^R) is a locally isometric cover of N_G.

Let m be an integer sufficiently large so that $B_m(\omega_G)$ contains C. Taking R in the integers greater than m, and repeating the above procedure, we obtain a family of sequences each with algebraic limit M whose geometric limits form a tower of covering spaces N_G^n, $n \geq m$.

We let $(N_G^\infty, \omega_G^\infty)$ be the geometric limit of the sequence of covers (N_G^n, ω_G^n). Diagonalizing, we have i_n so that $(M_{i_n}^n, \omega_{i_n}^n)$ converges geometrically to N_G^∞. Let $M_{i_n}^n = \hat{M}_n$ and let

$$\pi_n \colon (N_G^\infty, \omega_G^\infty) \to (N_G^n, \omega_G^n)$$

denote the locally isometric covering projection. We obtain virtually defined almost isometries

$$\psi_n \colon N_G^\infty \dashrightarrow \hat{M}_n$$

that converge C^∞ to an isometry on $K(\psi_n)$ so that ψ_n decomposes as the composition

$$\psi_n = \Psi_{i_n}^n \circ \pi_n$$

where $\Psi_m^n \colon (N_G^n, \omega_G^n) \dashrightarrow (M_m^n, \omega_m^n)$ are the virtually defined almost isometries coming from the convergence (M_m^n, ω_m^n) to (N_G^n, ω_G^n).

Let $\mathbb{P}$ be a cusp of N_G^∞ so that $B_R(\omega_G^\infty) \cap \mathbb{P}$ contains a horocyclic loop γ of length ℓ in $\mathbb{P}$. Then for n sufficiently large, π_n gives an isometric embedding of $B_R(\omega_G^\infty)$ to the ball $B_R(\omega_G^n)$ and $\Psi_{i_n}^n \circ \pi_n(\gamma) = \psi_n(\gamma)$ lies in a cusp of $\hat{M}_n$. It follows that the geometric convergence

$$(\hat{M}_n, \hat{\omega}_n) \to (N_G^\infty, \omega_G^\infty)$$

is geometrically type-preserving. $\square$

We conclude this section with the following consequences of theorems of [BBES], [AC2], and [Can3] which we record for future reference.

Proposition 2.5. (Opening Cusps) *Let $M_n \to M$ be an algebraically convergent sequence in $AH(N)$ that is geometrically type-preserving, with geometric limit $N_G = \mathbb{H}^3/\Gamma_G$ covered by M. Let $(\mathscr{M}_n, \mathscr{P}_n)$ be relative compact cores for M_n and let $\mathscr{P}_n'$ be any subset of the rank-one cusps of $\mathscr{P}_n$. Then there is a sequence M_n' in $AH(N)$ with the following properties:*

- *M_n' converges algebraically to M and geometrically to N_G,*

- *$CC(M_n')$ is homeomorphic to $\mathscr{M}_n \setminus \mathscr{P}_n'$, and*
- *if $\phi_n' \colon N_G \dashrightarrow M_n'$ are the resulting virtually defined almost isometries then for any tame subgroup $H < \Gamma_G$ on which the representations*

$$\rho_n = (\phi_n')_*|_H \colon H \to \pi_1(M_n')$$

are faithful, the restrictions ρ_n converge strongly. □

Proof. The proof of [BBES, Theorem 3.4] allows one to approximate each geometrically finite manifold strongly by geometrically finite manifolds with no rank-1 cusps; the argument opens all rank-1 cusps of the limit by promoting them to rank-2 using the combination theorem and applying Thurston's Dehn surgery theorem to fill them (see [Brm, Thm. 7.3], [BO]). This technique may just as easily be used to open cusps selectively; diagonalizing proves the first assertion. The second assertion is then evident.

Observe that by a diagonal argument it suffices to show the third assertion for each tame subgroup H. The assertion for the virtually defined almost isometries $\phi_n \colon N_G \dashrightarrow M_n$ follows from an application of [Can3, Theorem 9.2] if H has limit set $\widehat{\mathbb{C}}$, and from [AC2, Ev] when the limit set is not all of $\widehat{\mathbb{C}}$ after observing that the induced representations $(\phi_n)_*|_H$ are weakly type-preserving. Since the manifolds M_n' can be chosen in arbitrarily small neighborhoods of M_n in the strong topology, we may choose M_n' to ensure that the geometric limit of $(\phi_n)_*|_H(H)$ is the same as the geometric limit of $(\phi_n')_*|_H(H)$. The proposition follows. □

Notation. It will be at times convenient to work with a sequence (M_n', ω_n') some of whose cusps have been opened rather than the original geometrically type-preserving sequence (M_n, ω_n) as above. Since their qualitative features are essentially the same, we will avoid over-emphasizing the distinction between these two sequences. As such, given η parabolic in M_n but hyperbolic in M_n' we refer by $\mathbb{P}_\eta^\varepsilon(M_n')$ to the intersection $\mathbb{T}_\eta^\varepsilon(M_n') \cap CC(M_n')$. In the same spirit, we denote by $P^\varepsilon(M_n')$ the union of $\mathbb{P}_\eta^\varepsilon(M_n')$ (taken with this notational convention) over all η parabolic in M_n.

For such sequences M_n or their variants M_n', it will be convenient to extend the virtually defined almost isometric embeddings $\phi_n \colon N_G \dashrightarrow M_n$ from the compact subsets $K \subset N_G$ to proper embeddings of $K \cup P^\mu(N_G)$ taking cusps to cusps; we refer to such extensions as *extended almost isometric embeddings*.

3 A combination theorem

In this section we prove a combination theorem for tame manifolds that will play a central role in our argument. Let M be a hyperbolic 3-manifold with finitely generated fundamental group $\pi_1(M)$ and let C be a compact core of M.

Definition 3.1. *A properly embedded incompressible surface $\Sigma \subset M$ with finite topological type is said to be* peripheral *if it does not intersect the core C.*

Whether a surface is peripheral depends on the choice of compact core, but later we will give several characterizations of those properly embedded disks or incompressible annuli which are properly homotopic to a peripheral surface. We remark that peripheral surfaces separate since every closed curve M can be homotoped into the core C. Later we will need to consider peripheral surfaces which are not connected but we will assume without further reference that no component of a peripheral surface is separated from the core by a different component of the same surface. In other words, the peripheral surfaces Σ we consider are always contained in the closure of the component of $M \setminus \Sigma$ which contains the core.

Proposition 3.2. *Let Σ be a peripheral surface in M and assume that M is tame. Let V be the component of $M \setminus \Sigma$ containing the core C, let $U_1, \dots, U_k$ be the components of $M \setminus V$, and set $U = \cup_i U_i$.*

1. *The inclusions $V \hookrightarrow M$ and $\Sigma \hookrightarrow U$ are homotopy equivalences.*
2. *There is an exhaustion of M by nested compact cores C'_i such that the inclusion $C'_i \cap \Sigma \hookrightarrow \Sigma$ is a homotopy equivalence. Moreover $C'_i \cap V$ is a core of V for all i.*
3. *U is homeomorphic to $\Sigma \times \mathbb{R}_+$.*

We recall that a manifold can be exhausted by (relative) compact cores if there exists a sequence (C_i) of (relative) compact cores with $C_i \subset C_{i+1}$ and with $M = \cup_i C_i$. Before we start the proof of Proposition 3.2 we recall the following crucial observation:

Lemma 3.3. *Let $C_0 \subset C_1$ be compact cores of M, then C_0 is a compact core of C_1 and the surface ∂C_1 is incompressible and acylindrical in $M \setminus C_0$.* □

Proof of Proposition 3.2. We prove the first claim of Proposition 3.2. Let $\Sigma_1, \dots, \Sigma_k$ be the components of Σ, i.e. $\Sigma_i = \partial U_i$. From the Seifert-van Kampen theorem we deduce that

$$\pi_1(M) = \pi_1(V) *_{\pi_1(\Sigma_1)} \pi_1(U_1) *_{\pi_1(\Sigma_2)} \dots *_{\pi_1(\Sigma_k)} \pi_1(U_k) \tag{3.1}$$

The peripheral surface Σ is by definition incompressible, and this implies that $\pi_1(V)$ injects into $\pi_1(M)$; $\pi_1(V)$ surjects because V contains the core C. In particular, V is homotopy equivalent to M.

Moreover, we deduce that the action of $\pi_1(M)$ on the Bass-Serre tree associated to (3.1) has a global fixed-point. In particular, we obtain that the homomorphism $\pi_1(\Sigma_i) \to \pi_1(U_i)$ is an isomorphism. Proposition 3.2 (1) follows.

Before beginning the proof of (2) we establish the following lemma.

Lemma 3.4. *Let Σ be a properly embedded, finite type, incompressible surface in M and let C_1, C_2 be cores of M with $C_1 \subset C_2$. Assume that we have a subsurface $\Sigma_0 \subset \Sigma$ which is homotopy equivalent to Σ and such that*

$$C_1 \cap \Sigma \subset \Sigma_0 \subset C_2 \cap \Sigma.$$

Then there is a core C_2' in M_i with $C_1 \subset C_2'$ and with $\Sigma_0 = C_2' \cap \Sigma$.

Proof. Up to replacing C_2 by $\mathcal{N}_\delta(C_2) = \{x \in M | d(x, C_2) \leq \delta\}$ for some $\delta > 0$ we may assume that the intersection $\Sigma \cap C_2$ is a compact surface X with boundary $\partial X \subset \Sigma$. By assumption, the surface X contains Σ_0.

If some boundary component of X is compressible in $\Sigma \setminus \Sigma_0$, then there is a boundary component $\gamma \subset \partial X$ of X which bounds a disk $D \subset \Sigma \setminus \Sigma_0$ such that $D \cap \partial X = \gamma$.

Lemma 3.3 shows that there is a disk $D' \subset \partial C_2$ such that $D \cup D'$ is the boundary of a 3-cell B which either is contained in $C_2 \setminus C_1$ or in $M \setminus C_2$. If $B \subset C_2 \setminus C_1$, then set $\overline{C}_2 = C_2 \setminus \mathcal{N}_\delta(B)$ for $\delta > 0$ very small, else set $\overline{C}_2 = C_2 \cup \mathcal{N}_\delta(B)$. The submanifold $\overline{C}_2$ is isotopic in M to C_2, thus it is a core; further, we have $C_1 \subset \overline{C}_2$. Moreover, there are fewer curves in $\partial(\overline{C}_2 \cap \Sigma)$ that are compressible in $\Sigma \setminus \Sigma_0$ than in ∂X.

After finitely many repetitions of this process we obtain a core, (which we again call C_2) with $C_1 \subset C_2$ and such that all components of $\partial(C_2 \cap \Sigma)$ are incompressible in $\Sigma \setminus \Sigma_0$.

Let Σ_1 be the component of $\Sigma \cap C_2$ which contains Σ_0.

Suppose that $\Sigma \cap C_2 \setminus \Sigma_1$ is not empty and let A be one of its components. The surface A is an annulus which is properly embedded in $(C_2, \partial C_2)$ and which does not intersect C_1. Lemma 3.3 implies that there is an annulus $A' \subset \partial C_2$ with $\partial A' = \partial A$ such that $A \cup A'$ bounds a solid torus $T \subset C_2 \setminus C_1$. As above, there is $\delta > 0$ very small such that the set

$$\overline{C}_2 = C_2 \setminus \{x | d(x, T) < \delta\}$$

is isotopic to C_2 in M, contains C_1 and Σ_1 and the number of components of $\Sigma \cap \overline{C}_2$ is strictly smaller than that of $\Sigma \cap C_2$.

Repeating this process finitely many times we obtain a core which we call again C_2, which contains C_1, with $S_0 \subset C_2 \cap \Sigma$ and such that $\Sigma \cap C_2$ is homotopy equivalent to Σ. Now we can isotope C_2 in a small neighborhood of Σ to a core C_2' with the desired properties. $\square$

We continue with the proof of Proposition 3.2 (2). Let (Σ_i) be a sequence of compact subsurfaces of Σ which are homotopy equivalent to Σ, with $\Sigma_i \subset \Sigma_{i+1}$ and with $\Sigma = \cup_i \Sigma_i$.

Since M is assumed to be tame, there is an exhaustion by compact cores

$$C = C_0 \subset C_1 \subset \ldots.$$

There is some i_1 with $\Sigma_1 \subset C_{i_1}$. Lemma 3.4 shows that there is a core C_1' with $C_0 \subset C_1'$ and with $C_1' \cap \Sigma = \Sigma_1$. Now there is $i_2' \gg 1$ such that $C_1' \cup B_1(C_0) \subset C_{i_2'}$. Fix $n_2 \gg 1$ with $C_{i_2'} \cap \Sigma \subset \Sigma_{n_2}$ and let

$i_2 > i_2'$ be such that $\Sigma_{n_2} \subset C_{i_2} \cap \Sigma$. As above we find a core C_2' with $C_{i_2'} \subset C_2'$ and with $\Sigma_{n_2} = C_2' \cap \Sigma$. Proceeding inductively, we find an exhaustion of M by compact cores $C_0 \subset C_1' \subset C_2' \subset \dots$ with $C_j' \cap \Sigma = \Sigma_{n_j}$ for all $j \geq 1$.

It remains to prove that $C_j' \cap V$ is a core of V for all j. By construction, the core $C = C_0$ is contained in C_j' for all j. In particular, we deduce that the homomorphism $\pi_1(C_j' \cap V) \to \pi_1(V)$ is surjective. The surface $\Sigma \cap C_j'$ is incompressible in C_j', hence the homomorphism $\pi_1(C_j' \cap V) \to \pi_1(C_j') = \pi_1(M)$ is injective. The claim follows now from part (1).

Finally we prove Proposition 3.2 (3). Without loss of generality we may now assume that Σ, and hence U, is connected. Moreover, it follows from [Can2, Propositon 3.2] that the cover M_Σ of M corresponding to the group $\pi_1(\Sigma)$ is also tame. In particular, we deduce from (1) that the inclusion of U into M_Σ is a homotopy equivalence. Let C_i' be cores of M_Σ as in (2) and set $K_i = C_i' \cap U$ and $\Sigma_i = C_i' \cap \partial U \subset \partial K_i$. It follows from (2) that K_i is a core of M_Σ. In particular, $\partial K_i \setminus \Sigma_i$ is incompressible in K_i. We deduce that the pared manifold $(K_i, \partial \Sigma_i)$ is a (relative) compression body with incompressible boundary. This implies that $K_i = \Sigma_i \times [0,1]$ for all i. We deduce that U is homeomorphic to $\Sigma \times \mathbb{R}_+$. $\square$

Our strategy in the proof of Theorem 1.1 will be to decompose the algebraic limit into tame pieces along essential surfaces. This approach is motivated by the following result.

Theorem 3.5. *Let M be a complete hyperbolic 3-manifold with finitely generated fundamental group $\pi_1(M)$ and assume that $\Sigma \subset M$ is a properly embedded, incompressible, two-sided finite type surface. If for every component U of $M \setminus \Sigma$ the manifold $M_U = \mathbb{H}^3/\pi_1(U)$ is tame then M is tame.*

Proof. Let M and Σ be as in the statement of the theorem. By an induction argument, we can assume that Σ has only one component. We are going to show that M admits an exhaustion by nested compact cores. Tameness of M follows then from the main result of [Sou].

The complement $M \setminus \Sigma$ of Σ has either only one component U or two components U_1 and U_2. We treat only the second case, the former is analogous. Let M_{U_i} be the cover of M determined by U_i; U_i lifts homeomorphically to M_{U_i} and the embedding $U_i \hookrightarrow M_{U_i}$ is a homotopy equivalence. In particular, every core of U_i is a core of M_{U_i}. Choose a core C_{U_i} of U_i. The surface $\partial \overline{U}_i \subset M_{U_i}$ projects homeomorphically onto Σ and does not intersect the core C_{U_i}; in particular it is peripheral.

Let $K \subset M$ be any compact set and let $K_i = K \cap U_i$. By the proposition 3.2 there is a compact core C_i of M_{U_i} such that $K_i \subset U_i \cap C_i$, such that $C_i \cap U_i$ is a core of U_i, such that $C_i \cap \Sigma$ is homotopy equivalent to Σ and such that

$$C_1 \cap \partial \overline{U}_1 = C_1 \cap \Sigma \subset C_2 \cap \Sigma = C_2 \cap \partial \overline{U}_2.$$

This implies that $C_1 \cap C_2$ is the surface $C_1 \cap \partial \overline{U}_1$; the embedding of this surface into Σ is a homotopy equivalence.

Set $C = C_1 \cup C_2$; we have that $\pi_1(C) = \pi_1(C_1) *_{\pi_1(C_1 \cap C_2)} \pi_1(C_2)$. Further, we have that the following diagram commutes:

$$\begin{array}{ccc} \pi_1(C) & \longrightarrow & \pi_1(M) \\ \downarrow & & \downarrow \\ \pi_1(C_1) *_{\pi_1(C_1 \cap C_2)} \pi_1(C_2) & \longrightarrow & \pi_1(M_{U_1}) *_{\pi_1(\Sigma)} \pi_1(M_{U_2}) \end{array}$$

where the arrow below is given on each factor by the inclusion of C_i in M_{U_i}; hence it is an isomorphism. The vertical arrows are isomorphisms too, so we obtain that the homomorphism $\pi_1(C) \to \pi_1(M)$ is an isomorphism; we have proved that C is a core of M. By construction it contains the compact set K. As K is arbitrary, this implies that M admits a nested exhaustion by compact cores. Again the main result of [Sou] implies that M is tame. $\square$

We conclude this section with the following Lemma characterizing which disks and annuli are isotopic to peripheral surfaces.

Lemma 3.6. *Let M be a hyperbolic 3-manifold with finitely generated non-virtually abelian fundamental group, C a compact core of M and Σ a properly embedded disk or a properly embedded incompressible annulus. Then the following are equivalent:*

1. *Σ is properly homotopic to a peripheral surface.*
2. *There is a component V of $M \setminus \Sigma$ such that the inclusion $V \hookrightarrow M$ is a homotopy equivalence.*
3. *There is a component V of $M \setminus \Sigma$ such that the inclusion $\Sigma \hookrightarrow M \setminus V$ is a homotopy equivalence.*
4. *There is a component V of $M \setminus \Sigma$ with $M \setminus V = \Sigma \times \mathbb{R}_+$.*

A properly embedded disk or a properly embedded incompressible annulus for which one of these conditions fails is said to be essential.

Proof. The arguments in the proof of the first part of Proposition 3.2 show that the implications (1)⇒(2)⇒(3) hold.

The implication (3)⇒(4) follows when we apply Proposition 3.2 (3) to the cover of M determined by Σ; the key point is that the cover of M corresponding to Σ is tame because $\pi_1(\Sigma)$ is either trivial or abelian.

The implication (4)⇒(1) is obvious. $\square$

4 The proof of the main theorem

In this section we give the proof of the main theorem assuming the conclusion of Proposition 7.1 below, whose proof we defer to section 7.

Theorem 1.1. *Let M be an algebraic limit of geometrically finite hyperbolic 3-manifolds. Then M is tame.*

We remind the reader that after [BBES], the remark preceding Lemma 2.1, and Theorem 2.3, we may assume that the following condition holds here and in the sequel.

Working Assumption. *The hyperbolic 3-manifold M satisfies the following conditions: M has empty conformal boundary, M has a compact core C homeomorphic to a compression body, and M is the algebraic limit of a sequence (M_n) of geometrically finite manifolds that converge in a geometrically type-preserving manner to a limit N_G covered by M.*

In the remaining sections, we give an interpolation argument with simplicial hyperbolic surfaces to show the following key proposition which we take as our jumping off point for the proof of Theorem 1.1. The following proposition is proven in section 7.

Proposition 7.1. *Assume the geometric limit N_G is not a product. Then there is a properly embedded finite type surface $S \hookrightarrow N_G$ so that one component E of $N_G \setminus S$ is homeomorphic to $S \times \mathbb{R}_+$ and the surface S either is compressible or contains an essential simple closed curve γ that is homotopic into a cusp in N_G but not entirely within E.*

We now prove Theorem 1.1.

Proof of Theorem 1.1. It follows from Thurston's hyperbolization theorem that a manifold is tame if it has finitely generated fundamental group and covers a tame hyperbolic manifold of infinite volume (see [Can2, Proposition 3.2]). In particular M is tame if N_G is a product, i.e. N_G is homeomorphic to a trivial interval bundle over a compact surface. Otherwise, we satisfy the hypotheses of Proposition 7.1.

If the surface S given by Proposition 7.1 is compressible, let C_S be the relative compression body associated to $\partial(N_G \setminus E) = S$. By abuse of notation we denote also by C_S the submanifold $C_S \cup E$ which is homeomorphic to $C_S \setminus S$ by the product structure of E.

If C_S is a handlebody, then N_G, and hence M, is tame, so we consider the case that C_S has non-empty interior boundary $\partial_{\text{int}} C_S$. Let $\mathscr{D}$ be a collection of disks in C_S so that $C_S \setminus \mathscr{D}$ is homeomorphic to $\partial_{\text{int}} C_S \times \mathbb{R}$. By construction every component V of $N_G \setminus \mathscr{D}$ contains at least one component $\partial_V C_S$ of the interior boundary of C_S. Moreover, the surface $\partial_V C_S$ divides V and one of its two components, call it E_V, is homeomorphic to $\partial_V C_S \times \mathbb{R}$. As a component of the interior boundary of C_S, the surface $\partial_V C_S$ is incompressible in N_G and hence determines a tame end of the cover $N_V = \mathbb{H}^3/\pi_1(V)$ of N_G by Bonahon's theorem. In other words, the manifold V lifts homeomorphically to N_V and the complement of $V \setminus E_V$ in N_V is homeomorphic to $\partial_V C_S \times \mathbb{R}$.

The covering $N_V \to N_G$ has infinite degree because $\mathscr{D}$ is a non-empty collection of essential disks. Thus the restriction of this covering to $N_V \setminus (V \setminus E_V)$ cannot have finite fibers. By the covering theorem, we conclude that not all relative ends of N_V contained in the end $N_V \setminus (V \setminus E_V)$ can be degenerate. It follows that if V is a component of $N_G \setminus \mathscr{D}$ then the cover N_V of N_G corresponding to V has non-empty conformal boundary.

Now let $\mathscr{D}'$ be the preimage of $\mathscr{D}$ in the algebraic limit M. Only finitely many disks in $\mathscr{D}'$ intersect the core and, by definition, this implies that only finitely many disks in $\mathscr{D}'$ are essential. If a connected component $D \subset \mathscr{D}'$ is not essential then it must separate M, and thus a component of $M \setminus D$ is a ball by Lemma 3.6. The complement of the union of such balls is a submanifold of M, which we denote by $M_{\mathscr{D}}$, whose inclusion in M is a homotopy equivalence. We denote by $\mathscr{D}'_{\text{ess}} \subset \mathscr{D}'$ the union of disks in the subcollection consisting of those that are essential in M.

Let $U_1, \dots, U_l$ be the connected components of $M_{\mathscr{D}} \setminus \mathscr{D}'_{\text{ess}}$, and let $\widetilde{M}_j = \mathbb{H}^3/\pi_1(U_j)$. We remark that by construction each of the covers $\widetilde{M}_j$ is the algebraic limit of a sequence of geometrically finite groups and that the image $\pi(U_j)$ of the component U_j is contained in a component V_j of $N_G \setminus \mathscr{D}$. As we proved above, the conformal boundary of N_{V_j} is non-empty and thus that the conformal boundary of $\widetilde{M}_j$ is non-empty *a fortiori*. The main theorem of [BBES] implies that $\widetilde{M}_j$ is tame. Tameness of M follows now from successive applications of Theorem 3.5. This concludes the proof of Theorem 1.1 in the case that the surface S is compressible.

Assume now that S is incompressible. Then there is a curve γ in S which can be homotoped into a cusp of N_G but such that this homotopy cannot lie entirely within the tame end E. We face the difficulty that the cusp corresponding to γ could nevertheless be contained in E; we assume first that this is not the case. Then there is a properly embedded annulus $A \simeq \mathbb{R} \times S^1 \subset N_G$ with $A \cap E = \mathbb{R}_+ \times S^1$ and such that the core of A represents a parabolic element in N_G while its corresponding cusp does not intersect E. Replacing the collection $\mathscr{D}$ of disks in the above argument by the annulus A and replacing the preimages of $\mathscr{D}$ in the algebraic limit by the preimages of A (which may either be annuli or disks in this case), the proof may be carried out exactly as before.

Assume now that the cusp corresponding to γ lies within E. We choose a connected submanifold B of N_G that contains the end E and the core C and has finitely generated fundamental group. Let N_B be the cover of N_G determined by the image of $\pi_1(B)$ into $\pi_1(N_G)$. Since $\pi_1(N_B)$ is finitely generated, it is residually finite by Selberg's Lemma. Thus, there is a finite normal cover $N'_B \to N_B$ such that a component of the pre-image E' of E is a tame incompressible end of N'_B and such that the corresponding component S' of the pre-image of S contains a curve which can be homotoped into a cusp of N'_B that is not contained in E'.

By construction, the algebraic limit M covers N_B and therefore there is a finite cover M' of M which covers N'_B. The proof above shows that M' is tame. We claim that this implies that M is tame. If X is a compact submanifold of M with pre-image X' in M' then $\pi_1(M \setminus X)$ is a finite extension of $\pi_1(M' \setminus X')$ which is finitely generated since M' is tame and X' is compact. Therefore $\pi_1(M \setminus X)$ is finitely generated for any compact submanifold X. By a theorem of Tucker (see

[Tck]), we may conclude that M is tame. This concludes the proof of Theorem 1.1. $\square$

The remaining sections of the paper are devoted to the proof of Proposition 7.1.

5 Simplicial hyperbolic surfaces

Let $\bar{S}$ be a closed surface, $\mathscr{V} \subset \bar{S}$ a finite collection of points and $S = \bar{S} - \mathscr{V}$. A continuous map $\phi : S \to N$ into a hyperbolic 3-manifold is a *pre-simplicial hyperbolic surface* if the following conditions hold:

- The boundary of every small disk centered at a point in $\mathscr{V}$ is mapped by ϕ to an essential curve in N.
- There is a triangulation $\mathscr{T}$ of $\bar{S}$ which contains $\mathscr{V}$ in the set of vertices such that every face of $\mathscr{T}$ is mapped by ϕ to a totally geodesic triangle.

See Hatcher [Hat] for a precise definition of triangulation. We remark that it follows from the definition that every pre-simplicial hyperbolic surface is proper. A pre-simplicial hyperbolic surface S in N is a *simplicial hyperbolic surface* if the cone angle at each vertex of $\mathscr{T}$ that does not belong to $\mathscr{V}$ is at least 2π. If $\phi : S \to N$ is a simplicial hyperbolic surface we will often say that S itself is a simplicial hyperbolic surface in N. The distance induced on the universal cover of a simplicial hyperbolic surface S in N is complete and has curvature ≤ -1 in the sense of Alexandroff, hence [Bon, Can1] we have

$$\mathrm{vol}(S) \leq 2\pi|\chi(S)|.$$

In the cusp-free setting, Canary and Minsky showed that each component of the convex core boundary of a complete hyperbolic 3-manifold N admits an approximation by simplicial hyperbolic surfaces [CM]. The cusped case is similar.

Proposition 5.1. *Let $\mathscr{S}$ be a component of the boundary $\partial CC(N)$ of the convex core $CC(N)$ of a geometrically finite hyperbolic manifold N. Then for every $\varepsilon > 0$ there is a simplicial hyperbolic surface X which is properly homotopic to $\mathscr{S}$ by a homotopy whose tracks are all shorter than ε.*

Simplicial hyperbolic surfaces are useful because simple moves performed on their associated triangulations can be translated to homotopies through simplicial hyperbolic surfaces. A homotopy (X^t) with X^t simplicial hyperbolic for all t is said to be an *interpolation*. The following propostion is due to Canary [Can3, Section 5] and Canary-Minsky [CM, Proposition 4.5].

Proposition 5.2. *Let N be a complete hyperbolic* 3*-manifold and let X be a simplicial hyperbolic surface in N:*

1. *If X is incompressible with no accidental parabolics and X' is a second simplicial hyperbolic surface in N homotopic to X, then there is an interpolation $(X^t)_{t\in[0,1]}$ with $X^0 = X$ and $X^1 = X'$.*
2. *Suppose that X is either compressible or has accidental parabolics or both. Then, there is for every $\varepsilon > 0$ an interpolation $(X^t)_{t\in[0,1]}$ with $X = X^0$ such that $l_{X^1}(\gamma) < \varepsilon$ for some essential non-peripheral curve γ which is either compressible or represents a parabolic element in N.*

Recall that an essential non-peripheral simple closed curve γ on a surface X in a hyperbolic manifold N is an *accidental parabolic* if it can be homotoped into a cusp of N.

The following estimates on the geometry of simplicial hyperbolic surfaces are probably well-known. We include them for completeness.

Lemma 5.3. *There is an $\varepsilon > 0$ such that if X is a simplicial hyperbolic surface in a hyperbolic manifold N and $\eta \subset X$ is a simple closed curve with $\ell_X(\eta) \leq \varepsilon$ for which η intersects every compressible curve in X essentially, then the set $\{x \in X | d_X(x,\eta) \leq 1\}$ is an embedded annulus.*

Proof. It suffices to show that every component A of $\{d_X(\cdot,\eta) \leq 1\} \setminus \eta$ is an annulus. We choose ε sufficiently small such that A is contained in a component of the μ-thin part of N. In particular, the image of $\pi_1(A)$ into $\pi_1(N)$ is abelian. Convexity of the distance function on the universal cover of S guarantees that if A fails to be an annulus then the image of $\pi_1(A)$ into $\pi_1(X)$ contains a free group. We deduce that there is a curve γ in A which is essential in X but compressible in N contradicting the assumption on η. □

Now let X be a simplicial hyperbolic surface in a hyperbolic 3-manifold M and let $\varepsilon < \mu$ be such that every compressible curve in X has length at least 4ε. Observe that every component A of $X^{<\varepsilon}$, ε-thin part of X, is contained in a component of $N^{<\varepsilon}$. If A fails to be an embedded annulus then there is a point $x \in A$ and two loops γ and η based at x with length less than ε so that γ and η generate a free subgroup of $\pi_1(X)$. As in the proof of Lemma 5.3 we have that both elements commute in $\pi_1(N)$ and hence their commutator $[\gamma,\eta]$ is essential in X, compressible in N and has length less than 4ε; this yields a contradiction, resulting in the following.

Lemma 5.4. *For all $\varepsilon > 0$ there is an $\varepsilon' > 0$ such that for every simplicial hyperbolic surface X in a hyperbolic manifold N with all compressible curves of length at least ε, the ε'-thin part of X has at most $\frac{3}{2}|\chi(X)|$ components and each component is an annulus.* □

Finally, we establish the following uniform relative diameter bound.

Lemma 5.5. *Let (M_n, ω_n) be a sequence of framed hyperbolic manifolds which converges geometrically type-preserving to a manifold (N_G, ω_G). For every d, ε and A there is a constant D such that for every sequence (X_n) of simplicial hyperbolic surface with $|\chi(X_n)| \leq A$, with $X_n \cap B_d(\omega_n) \neq \emptyset$ and with $l_{X_n}(\gamma) \geq \varepsilon$ for every essential curve $\gamma \subset X_n$ which is either compressible or homotopic into $P(M_n)$ then we have $X_n \subset B_D(\omega_n) \cup P^\varepsilon(M_n)$.*

Proof. To begin with we choose δ to be less than ε, the Margulis constant μ and the constant provided by Lemma 5.4.

The claim of the lemma follows if we can bound independently of n the diameter of the complement $X_n \setminus P^\delta(X_n)$ of the cuspidal part $P^\delta(X_n)$ of X_n, where the cuspidal part is the union of all unbounded components of $X_n^{<\delta}$ (recall we will use the notation $P^\delta = P^\delta(X_n)$ when there is no danger of confusion). Seeking a contradiction assume that this is not the case and fix for all n points $x_n \in X_n \cap B_d(\omega_n)$ and $y_n \in X_n \setminus P^\delta$ such that $d_{X_n}(x_n, y_n) \to \infty$ when $n \to \infty$. Choose also a minimal geodesic $\gamma_n : [0, d(x_n, y_n)] \to X_n$ and note that the length of $\gamma_n[0, d_{X_n}(x_n, y_n)] \cap P^\delta$ is bounded from above by a constant depending on δ and on the number of components of P^δ because γ_n minimizes length.

Let $I_1, \ldots, I_k$ be the segments in $[0, d(x_n, y_n)]$ which parametrize the crossings of γ_n with the components of the δ-thin part of X_n; the minimality of γ_n implies that every component is crossed at most once. We set $I_j = [t_n^j, s_n^j]$ with $t_n^1 < s_n^1 < t_n^2 < s_n^2 < \cdots < t_n^k < s_n^k$; note that $\gamma_n(s_n^k) \in \varepsilon_5 - \operatorname{thin}(X_n)$ if $s_k^i = d(x_n, y_n)$. The estimate $\operatorname{vol}(X_n) \leq 2\pi|\chi(X_n)| \leq 2\pi A$ and a simple volume comparison argument [Th1, Bon] imply that there is some uniform constant B with

$$d(x_n, y_n) - \sum_{j=1}^{k} |s_n^j - t_n^j| \leq B$$

for all n. In particular, there is some m with $\lim_n |s_n^j - t_n^j| = \infty$; we take m minimal with this property. Thus $(t_n^j)_n$ is bounded, say convergent to $t_\infty \in \mathbb{R}$ after passing to a subsequence, and $\lim_n s_n^j = \infty$. The lower bound on the length of compressible curves together with a comparison argument shows that for all $\tau \in (t_n^j, \frac{1}{2}(t_n^j + s_n^j))$ we have

$$\operatorname{inj}_{M_n}(\gamma_n(\tau)) \leq \operatorname{inj}_{X_n}(\gamma_n(\tau)) \leq \frac{\delta}{\cosh(\tau - t_n)}. \tag{5.2}$$

Up to passing to a subsequence, we may assume that the maps $\phi_n^{-1} \circ \gamma_n$ converge to a 1-Lipschitz map $\gamma_\infty : [0, \infty) \to N_G$ with $\gamma_\infty(0) \in B_d(\omega_G)$. Moreover, we have that $\gamma_\infty(t_\infty, \infty)$ is contained in a single component V of the δ-thin part of N_G which is unbounded by (5.2).

The uniform length decay property of the sequence (M_n) implies that the component of the δ-thin part of M_n containing $\gamma_n\left(\frac{1}{2}\left(t_n^j + s_n^j\right)\right)$ is unbounded for all n, which implies that that the core curve of the corresponding component of the δ-thin part of X_n is an accidental parabolic. Equation (5.2) contradicts the lower bound for the length of those curves. $\square$

We remark that Lemma 5.5 readily applies to sequences X_n taken in filled manifolds M_n' obtained from Proposition 2.5 with the notational convention following the proof of the Proposition. (Note that this application fails without the notation convention).

6 Pulling down

In this section we employ the geometrically type-preserving assumption to generalize the interpolation arguments of Canary and Minsky [CM]. We remind the reader that we argue always under the *working assumption* of section 4. We also let $C \subset M$ be a compact core that embeds in N_G (whose existence is guaranteed by Lemma 2.1) and let $C_n = \phi_n(C) \subset M_n$ be compact cores obtained as the images of C under the almost isometries $\phi_n \colon N_G \dashrightarrow M_n$. In this section, we will prove the following.

Proposition 6.1. *Either the geometric limit N_G is a product, or there is a properly embedded finite type surface $S \hookrightarrow N_G$ so that one component E of $N_G \setminus S$ is homeomorphic to $S \times \mathbb{R}_+$ and such that*

- *the virtually defined almost isometries $\phi_n \colon N_G \dashrightarrow M_n$ extend to proper embeddings ϕ_n^E of E, and*
- *$\phi_n(S)$ is either compressible or contains an essential curve which is homotopic into a cusp of M_n but not entirely within $\phi_n^E(E)$.*

Before beginning the proof, we fix once and for all a positive constant ε_0 less than the Margulis constant and the constants provided by Lemma 5.3 and Lemma 6.2 below, and such that

$$C_n \cap M_n^{<\varepsilon_0} = \emptyset$$

for all n.

Lemma 6.2. *There is $\varepsilon > 0$ such that for all n we have $\ell_{M_n}(\gamma) > \varepsilon$ for each curve $\gamma \subset M_n \setminus C_n$ which is either compressible or homotopic into a cusp in M_n but not in $M_n \setminus C_n$.*

Proof. If γ is compressible it bounds a necessarily non-embedded, ruled disk D. In particular the length of γ is bigger than the area of D. The disk D intersects the core C_n essentially. Since the cores C_n are chosen in a uniform way, there is a uniform lower bound to the area of every such essential disk; this yields the desired uniform bound. If γ can be homotoped into a cusp, we can realize this homotopy by a ruled annulus and the same argument applies. □

We pause to describe the basic strategy of the proof of Proposition 6.1. We realize the surface S as a limit of embedded surfaces in M_n, each of which is obtained from an interpolation of simplicial hyperbolic surfaces using results of section 5. In the case that the convex core boundary of M_n has a compressible component, Propositions 5.2 and 5.5 allow us to interpolate from the boundary of the convex core in to surfaces that converge geometrically after passing to a subsequence. These surfaces can be obtained by an interpolation that does not cross the core, and we may take a nearby embedded convergent representatives of the approximates and of the limit. The same method locates such embedded surfaces farther and farther from the basepoint, and methods of [Sou] provide that their limits give an exhaustion of a tame end of N_G.

A variant of this method using Bonahon's theorem treats the case when $\partial CC(M_n)$ contains a component with an accidental parabolic. Unfortunately, we cannot always assume $CC(M_n)$ contains such boundary components, so to apply methods of section 5 we open certain of the cusps of M_n to obtain manifolds for which the same arguments can be fruitfully carried out.

Remark: We note that in either case the restrictions of the geometric limit mappings

$$\phi_n|_S \colon S \to M_n$$

that limit to the identity mapping id_S can range in infinitely many different homotopy classes with respect to markings on M_n. Indeed, this possibility represents a central difficulty in our argument, and represents the primary reason for seeking first a tame end of the *geometric* limit rather than considering the algebraic limit alone.

The proof of Proposition 6.1 divides into three cases depending on the properties of the pared manifold $(\mathscr{M}_n, \mathscr{P}_n)$ corresponding to M_n. The sequence (M_n) will be replaced in some cases by a sequence (M_n') with associated pared manifold $(\mathscr{M}_n, \hat{\mathscr{P}}_n)$ with $\hat{\mathscr{P}}_n \subset \mathscr{P}_n$ as provided by Proposition 2.5. Recall that (by our notational convention following Proposition 2.5) given a curve η homotopic into a rank-1 cusp of M_n, we abuse notation and denote by $\mathbb{P}^{\varepsilon_0}_\eta = \mathbb{P}^{\varepsilon_0}_\eta(M_n')$ the intersection of the component of the ε_0-thin part corresponding to η of M_n' with the convex core of M_n'. Similarly, we denote by $P^{\varepsilon_0}(M_n')$ the union of the cuspidal ε_0-thin part of M_n' with all components $\mathbb{P}^{\varepsilon_0}_\eta$ where η corresponds to components of $\check{\mathscr{P}}_n = \mathscr{P}_n \setminus \hat{\mathscr{P}}_n$. By construction of the manifolds M_n', for each η corresponding to a component of $\check{\mathscr{P}}_n$ we have $\partial\mathbb{P}^{\varepsilon_0}_\eta$ is an annulus where $\partial\mathbb{P}^{\varepsilon_0}_\eta$ is the intersection of the usual topological boundary of the η-Margulis tube $\mathbb{T}^{\varepsilon_0}_\eta$ with the interior of the convex core of M_n'.

We have moreover almost isometric maps guaranteed by Proposition 2.5 pushing forward the core C_n of M_n we obtain a core of M_n' which we denote again by C_n. These almost isometric maps can be extended to homeomorphisms; thus Proposition 6.1 follows if we prove the same claims for the sequence (M_n'). We may moreover assume that for all n we have $\ell_{M_n}(\gamma) > \varepsilon_0$ for each curve $\gamma \subset M_n' \setminus C_n$ which is either compressible or homotopic into $P^{\varepsilon_0}(M_n')$ in M_n' but not in $M_n' \setminus C_n$.

Notation. The relevant data recorded by the parabolic locus $\mathscr{P}_n$ are the homotopy classes of closed curves in M_n that represent parabolic elements of $\pi_1(M_n)$. We warn the reader that we will frequently abuse the distinction between an essential curve $\eta \subset \mathscr{P}_n$ and its free homotopy class.

We now commence the proof of Proposition 6.1. As a warm-up we consider the following special case; many of its arguments are repeated later in the proof.

Case 1: *There is an identification $(\mathscr{M}_n, \mathscr{P}_n) \simeq (S_n \times [-1,1], \partial S_n \times [-1,1] \sqcup \mathscr{P}_n^+ \sqcup \mathscr{P}_n^-)$ where S_n is a compact surface, $\mathscr{P}_n^\pm \subset S_n \times \{\pm 1\}$ and such that $\mathscr{P}_n^+ \neq \emptyset \neq \mathscr{P}_n^-$ and representatives in S_n of each component $\mathscr{P}_n^+$ and each component of $\mathscr{P}_n^-$ intersect essentially.*

Let (M_n') be the sequence of geometrically finite manifolds with associated pared manifold $(S_n \times [-1,1], \partial S_n \times [-1,1])$ provided by Proposition 2.5 and denote by $\partial^\pm CC(M_n')$ the two components of the boundary of the convex core of M_n'. There are, by Proposition 5.1, simplicial hyperbolic surfaces $X_n^{\pm 1}$ homotopic to $\partial^{\pm} CC(M'_n)$ by a homotopy with very short tracks. In particular we may assume that

$$\max_{\eta \subset \mathscr{P}_n^{\pm}} l_{X_n^{\pm 1}}(\eta) \leq \varepsilon_0$$

The surfaces $X_n^{\pm 1}$ are incompressible and have no accidental parabolics and therefore we obtain from Proposition 5.2 an interpolation $(X_n^t)_{t\in[-1,1]}$ joining X_n^{-1} and X_n^1 by simplicial hyperbolic surfaces. By the assumption that the components of $\mathscr{P}_n^{\pm}$ intersect each other and Lemma 5.3 we have

$$\min_{\eta \subset \mathscr{P}_n^{-}} l_{X_n^1}(\eta) \geq 1.$$

In particular there is t_n with $\min_{\eta\subset\mathscr{P}_n^-} l_{X_n^{t_n}}(\eta) = \varepsilon_0$; set $X_n = X_n^{t_n}$ and let η_n be a curve in X_n which can be homotoped into $\mathscr{P}_n^-$ with $l_{X_n}(\eta_n) = \varepsilon_0$.

Lemma 6.3. *There is a $D > 0$ such that for all n there is a properly embedded surface $S_n \subset B_D(\omega_n) \cup P^{\varepsilon_0}(M'_n)$ which does not intersect $\mathrm{int}(\mathbb{P}_{\eta_n}^{\varepsilon_0})$, such that $S_n \cup \partial^- CC(M'_n)$ bound a product region and such that $S_n \cap \partial \mathbb{P}_{\eta_n}^{\varepsilon_0}$ contains a curve η_{S_n} homotopic to η_n within $\mathbb{P}_{\eta_n}^{\varepsilon_0}$.*

Proof. Since the surface X_n is incompressible and homotopic to an embedded surface, a theorem of Freedman-Hass-Scott [FHS] produces an embedded surface which is close to X_n. For these surfaces to be of use to us, however, we must show that they may be taken to lie within a uniform ball at ω_n; after our construction of the surface S_n most of the proof will be devoted verifying that they may be so taken. As a first step we observe that the diameter of $X_n \cap \partial \mathbb{P}_{\eta_n}^{\varepsilon_0}$ is bounded by $2|\chi(X_n)|/\varepsilon_0^2$; hence it is contained in an annulus $\bar{A}_n \subset \mathbb{P}_{\eta_n}^{\varepsilon_0}$ whose diameter is bounded by the same constant.

Let η'_n be the geodesic in X_n corresponding to η_n. The curves η_n and η'_n are homotopic in $\mathbb{P}_{\eta_n}^{\varepsilon_0}$; choose a cylinder a_n realizing this homotopy. Cutting open X_n along η'_n and gluing a_n to the boundary curves so obtained, we get a surface Z_n which is properly homotopic to the embedded incompressible surface $\partial^- CC(M'_n) \setminus \eta'_n$. In particular, applying [FHS] there is a properly embedded surface Z'_n contained in $\mathscr{N}_1(Z_n)$, properly homotopic to $\partial^- CC(M'_n) \setminus \eta'_n$ and with $Z'_n \cap \partial \mathbb{P}_{\eta_n}^{\varepsilon_0} \subset \bar{A}_n$. Up to slightly perturbing Z'_n we can assume that $Z'_n \cap \partial \mathbb{P}_{\eta_n}^{\varepsilon_0}$ is a finite collection of simple disjoint curves. These curves are either compressible or homotopic to η_n. In particular, there is a well-defined compact subsurface $Z''_n \subset Z'_n$ such that $\pi_1(Z''_n)$ surjects onto $\pi_1(Z'_n)$, whose boundary consists of two curves in $Z'_n \cap \partial \mathbb{P}_{\eta_n}^{\varepsilon_0}$ and such that every other component of $Z''_n \cap \partial \mathbb{P}_{\eta_n}^{\varepsilon_0}$ is compressible.

Arguing essentially as in the proof of Lemma 3.4, we can isotope Z''_n fixing its boundary to an embedded surface $\bar{Z}_n$ contained in a small neighborhood of $Z''_n \cup \bar{A}_n$ whose interior does not intersect $\partial \mathbb{P}_{\eta_n}^{\varepsilon_0}$. The boundary curves of $\bar{Z}_n$ bound a compact annulus $A_n \subset \bar{A}_n$ and we set $S_n = \bar{Z}_n \cup A_n$. Observe that the bound on the diameter of $\bar{A}_n$ implies that there is D_1 with $S_n \subset \mathscr{N}_{D_1}(X_n)$ for all n.

The surfaces S_n and $\partial^- CC(M'_n)$ are disjoint, incompressible and homotopic, and hence bound a product region [Wald]. Moreover, S_n does not intersect $\mathrm{int}(\mathbb{P}_{\eta_n}^{\varepsilon_0})$ and $S_n \cap \partial \mathbb{P}_{\eta_n}^{\varepsilon_0}$ contains a curve

η_{S_n} homotopic to η_n within $\mathbb{P}^{\varepsilon_0}_{\eta_n}$. It remains to show that the surfaces S_n are contained in a ball of uniform radius centered at ω_n.

We show first that S_n intersects a ball of uniform radius. If S_n does not intersect C_n then the assumption that $C_n \cap M_n^{<\varepsilon_0} = \emptyset$ implies that η_{S_n} can be homotoped into η_n without traversing the core. This implies that S_n separates the core from $\mathbb{P}^{\varepsilon_0}_{\eta_n}$. By the geometrically type-preserving assumption, the distance between C_n and $\mathbb{P}^{\varepsilon_0}_{\eta_n}$ is bounded; in particular, there is some D_2 such that $S_n \cap B_{D_2}(\omega_n) \neq \emptyset$ for all n. This implies that the simplicial hyperbolic surface X_n intersects the ball $B_{D_1+D_2}(\omega_n)$ for all n; X_n is incompressible and the choices of ε_0 and of X_n itself ensure that for every curve $\eta \subset X$ homotopic into $\mathscr{P}_n^+ \cup \mathscr{P}_n^-$ we have $l_X(\eta) \geq \varepsilon_0$. We deduce then from Lemma 6.3 and Lemma 5.5 that there is a constant D_3 with $X_n \subset B_{D_3}(\omega_n) \cup P^{\varepsilon_0}(M'_n)$ for all n. The claim follows when we set $D = D_3 + D_1$. $\square$

We can now conclude the proof of Proposition 6.1, always under the assumption that we are in Case 1. To begin with, we choose an irreducible and atoroidal submanifold V of N_G which contains $B_{2D}(\omega_G) \cup P^{\varepsilon_0}(N_G)$, with $V \setminus P^{\varepsilon_0}(N_G)$ compact. For n large enough, the extended almost isometric embedding ϕ_n (see section 2) is defined on V and its image contains the embedded surface S_n provided by Lemma 6.3. By construction, each of the surfaces S_n is incompressible and embedded. This implies that, up to passing to a subsequence, the surfaces $\phi_n^{-1}(S_n)$ and $\phi_m^{-1}(S_m)$ are homotopic for all m and n since V is homeomorphic to the interior of an irreducible and atoroidal compact manifold and such manifolds contain only finitely many homotopy classes of properly embedded surfaces with Euler-characteristic less than $\chi(\partial C)$; set $S = \phi_1^{-1}(S_1)$.

The cover $\mathbb{H}^3/\pi_1(S)$ of N_G determined by S is tame by Bonahon's theorem [Bon] and the surface $\phi_n(S)$ is incompressible for all n. This implies that the representations of $\pi_1(S)$ induced by $(\phi_n)_*|_{\pi_1(S)}$ converge strongly to the representation given by the inclusion of $\pi_1(S)$ into $\pi_1(N_G)$, by Proposition 2.5. In particular, the convex cores of the manifolds $M'_n = \mathbb{H}^3/(\phi_n)_*(\pi_1(S))$ converge to the convex core of $\mathbb{H}^3/\pi_1(S)$ (see, e.g., [KT, Kl]).

By assumption, the algebraic limit M has empty domain of discontinuity which implies that $\partial CC(M'_n)$ is farther and farther away from the base-frame ω_n. We deduce that $\mathbb{H}^3/\pi_1(S)$ has empty conformal boundary and thus each of its relative ends is degenerate. Therefore, the covering $\mathbb{H}^3/\pi_1(S) \to N_G$ is trivial by the covering theorem, after an application of [JM, Lemma 3.6] as before. We have proved that the geometric limit is homeomorphic to $S \times \mathbb{R}$ and thus Proposition 6.1 follows if we are in Case 1.

In the case that $(\mathscr{M}_n, \mathscr{P}_n) = (S \times [0,1], \partial S \times [0,1])$ we simply choose any simplicial hyperbolic surface that intersects the core. The rest of the proof remains the same.

Case 2: Every component of $\partial CC(M_n)$ is quasi-Fuchsian.

We may choose a maximal subcollection $\hat{\mathscr{P}}_n$ of $\mathscr{P}_n$ such that a component $\mathscr{S}_n$ of $\partial \mathscr{M}_n \setminus \hat{\mathscr{P}}_n$ is either compressible or contains an essential curve which can be homotoped to a curve in $\mathscr{P}_n$ in $\mathscr{M}_n$ but not in $\mathscr{S}_n$ and set $\check{\mathscr{P}}_n = \mathscr{P}_n \setminus \hat{\mathscr{P}}_n$. Let (M'_n) be the sequence of geometrically finite hyperbolic manifolds with associated pared manifold $(\mathscr{M}_n, \hat{\mathscr{P}}_n)$ provided by Proposition

2.5 and identify $\mathscr{S}_n$ with a component, which we also denote by $\mathscr{S}_n$, of $\partial CC(M'_n)$. There is by Proposition 5.1 a simplicial hyperbolic surface X_n^0 in M'_n which is properly homotopic to $\mathscr{S}_n$ by a homotopy whose tracks are all very short and Proposition 5.2 yields an interpolation $(X_n^t)_{t\in[0,1]}$ beginning in X_n^0 and such that there is an essential curve $\gamma \subset \mathscr{S}_n$ which is either compressible or homotopic to $\hat{\mathscr{P}}_n$ with $l_{X_n^1}(\gamma) \leq 1$. From the assumption that ε_0 is less than the constant provided by Lemma 5.3 and from the fact that X_n^0 is close to the boundary of the convex core we obtain

$$\max_{\eta\subset\check{\mathscr{P}}_n} l_{X_n^0}(\eta) \leq \varepsilon_0, \quad \min_{\eta\subset\check{\mathscr{P}}_n} l_{X_n^1}(\eta) \geq \varepsilon_0$$

As above, we find t_n with $\min_{\eta\subset\check{\mathscr{P}}_n} l_{X_n^{t_n}}(\eta) = \varepsilon_0$; set $X_n = X_n^{t_n}$ and choose $\eta_n \subset \check{\mathscr{P}}_n$ with $l_{X_n}(\eta_n) = \varepsilon_0$. We claim

Lemma 6.4. *Either we are in Case 1 or there is $D > 0$ such that for all n there is a properly embedded surface $S_n \subset B_D(\omega_n) \cup P^{\varepsilon_0}(M'_n)$ which does not intersect $\text{int}(\mathbb{P}^{\varepsilon_0}_{\eta_n})$, such that $S_n \cup \mathscr{S}_n$ bound a product region and such that $S_n \cap \partial\mathbb{P}^{\varepsilon_0}_{\eta_n}$ contains a curve η_{S_n} homotopic to η_n within $\mathbb{P}^{\varepsilon_0}_{\eta_n}$.*

Proof. Not only the statement but also the proof of this lemma is very similar to the proof of the Lemma 6.3. We only sketch the proof, pointing out the differences.

The construction of the embedded surface S_n is exactly as above: Gluing a cylinder between $\eta_n \subset \check{\mathscr{P}}_n$ and the corresponding geodesic η'_n in X_n to $X_n \setminus \eta'_n$ we obtain a proper surface Z_n which is incompressible and homotopic to an embedded surface Z'_n. We then obtain S_n by gluing the two cusps of Z'_n corresponding to η_n together. As above we obtain that S_n does not intersect $\text{int}(\mathbb{P}^{\varepsilon_0}_{\eta_n})$ and $S_n \cap \partial\mathbb{P}^{\varepsilon_0}_{\eta_n}$ contains a curve η_{S_n} homotopic to η_n within $\mathbb{P}^{\varepsilon_0}_{\eta_n}$. However, since the surface S_n may be compressible we cannot apply directly the Waldhausen co-bordism theorem to S_n and $\mathscr{S}_n$. We apply it instead to the two surfaces Z'_n and $\mathscr{S}_n \setminus \eta_n$; Z'_n and S_n bound a product by construction so we can conclude that S_n and $\mathscr{S}_n$ also bound a product. As above, it remains to show that the surfaces S_n are contained in a ball of uniform radius centered at ω_n. The proof of this claim is exactly as in Lemma 6.3 if the surface S_n either intersects the core or separates C_n from $\mathbb{P}^{\varepsilon_0}_{\eta_n}$.

It remains to consider the case that S_n fails to separate the core from $\mathbb{P}^{\varepsilon_0}_{\eta_n}$; then neither does the surface Z'_n. Hence Z'_n and $\mathscr{S}_n \setminus \eta'_n$ bound a product region containing C_n. In particular there is a component Y_n of Z'_n such that the inclusion $Y_n \hookrightarrow M_n$ is a homotopy equivalence and maps every curve which is boundary parallel to a parabolic in N_G. The minimality assumption on $\hat{\mathscr{P}}_n$ implies that we are in Case 1. $\square$

We continue the proof of Proposition 6.1 in Case 2. We choose again a submanifold V of N_G containing $B_{2D}(\omega_G) \cup P^{\varepsilon_0}(N_G)$ and consider the surfaces $\phi_n^{-1}(S_n)$. By Lemma 6.4, each of the surfaces S_n contains a simple closed curve η_{S_n} which can be isotoped inside of $\mathbb{P}^{\varepsilon_0}_{\eta_n}(M'_n)$ to the curve $\eta_n \subset \check{\mathscr{P}}_n$; in particular this isotopy does not cross S_n. This implies that the curve $\phi_n^{-1}(\eta_{S_n})$ represents a parabolic element in N_G and, moreover, can be isotoped into a cusp without crossing

$\phi_n^{-1}(S_n)$. Denote by a_n^G the annulus we obtain and let Z_n^G be the surface obtained by gluing two copies of a_n^G to the surface $\phi_n^{-1}(S_n \setminus \eta_{S_n})$. The surfaces Z_n^G are, after a small perturbation, properly embedded and incompressible in N_G. We deduce that as in Case 1, we may pass to a subsequence so that Z_n^G and Z_m^G are homotopic for all m and n. This implies in turn that the surfaces $\phi_n^{-1}(S_n)$ and $\phi_m^{-1}(S_m)$ are homotopic for all m and n because the cusp of N_G corresponding to the curve $\phi_n^{-1}(\eta_{S_n})$ has rank one. Set $S = \phi_1^{-1}(S_1)$, $\eta = \phi_1^{-1}(\eta_{S_1})$ and $Z = Z_1^G$. Note that the surfaces S and Z separate N_G because the surfaces S_n separate M_n'.

The cover $\mathbb{H}^3/\pi_1(Z)$ of N_G given by the surface Z is tame by Bonahon's theorem. In particular, the lift, also denoted by Z, of the surface Z to this cover separates $\mathbb{H}^3/\pi_1(Z)$ into two pieces which are homeomorphic to $Z \times \mathbb{R}$. The representations of $\pi_1(Z)$ induced by $(\phi_n)_*|_{\pi_1(Z)}$ are faithful, so Proposition 2.5 guarantees that they converge strongly to the representation given by the inclusion of $\pi_1(Z)$ into $\pi_1(N_G)$. By construction, the surface $\phi_n(Z)$ is properly isotopic to $\mathscr{S}_n \setminus \eta_n$. Since the discontinuity domain of the geometric limit N_G is empty we obtain that $\mathscr{S}_n \setminus \eta_n$ is farther and farther away from $\phi_n(Z)$.

In particular, there is a component F of $(\mathbb{H}^3/\pi_1(Z)) \setminus Z$ with the property that all the relative ends contained in F are degenerate. The covering theorem implies that F embeds under the covering $\mathbb{H}^3/\pi_1(Z) \to N_G$; we denote its image again by F. By construction, the almost isometric maps $\phi_n : N_G \dashrightarrow M_n'$ can be extended to maps ϕ_n^F which are defined on F such that $\phi_n^F|_F$ is a proper embedding and such that we have $\mathscr{S}_n \setminus \eta_n \subset \phi_n^F(F)$.

Let E be the component of $N_G \setminus S$ which contains F. Every relative end of N_G contained in E is by construction also contained in F and hence it is tame. This implies that E is homeomorphic to $S \times \mathbb{R}_+$. The almost isometric embeddings ϕ_n extend over E to maps ϕ_n^E so that $\phi_n^E|_E$ is a proper embedding.

This concludes the proof of Proposition 6.1 in Case 2 and we consider Case 3.

Case 3: For all n there is a component $\mathscr{S}_n$ of $\partial CC(M_n)$ which is not quasi-Fuchsian.

As above, we identify $\mathscr{S}_n$ with a component, which we also denote $\mathscr{S}_n$, of the boundary $\partial CC(M_n)$ of the convex core $CC(M_n)$ of M_n. First we are going to show that for all $R > 0$ there are n_R and $R' > 0$ such for all $n \geq n_R$ there is a simplicial hyperbolic surface X_n^R in M_n which is contained in $(B_{R'}(C_n) \cup P^{\varepsilon_0}(M_n)) \setminus B_R(C_n)$ and which is properly homotopic in $M_n \setminus B_R(C_n)$ to $\mathscr{S}_n$.

We begin fixing $R > 0$. Since we are only considering the case that the algebraic limit has empty conformal boundary we have n_R with $\mathscr{S}_n \cap B_R(\omega_n) = \emptyset$ for all $n \geq n_R$. Therefore we have a simplicial hyperbolic surface X_n^0 which is properly homotopic to $\mathscr{S}_n$ in $M_n \setminus B_R(C_n)$ by Proposition 5.1.

By Lemma 5.3 and Proposition 5.2 there is an interpolation $(X_n^t)_{t\in[0,1]}$ beginning at X_n^0 and with $X_n^1 \cap C_n = \emptyset$. Let t_n^R be minimal with $X_n^{t_n^R} \cap B_R(\omega_n) \neq \emptyset$ and set $X_n^R = X_n^{t_n^R}$. Lemma 6.2 implies that $l_{X_n^R}(\gamma) \geq \varepsilon_0$ for every essential curve $\gamma \subset \mathscr{S}_n$ which is either compressible or can be homotoped into a component of $\mathscr{P}_n$. Lemma 5.5 implies that there is D_3^R with $X_n^R \subset B_{D_3^R}(\omega_n) \cup P^{\varepsilon_0}(M_n)$ for all n.

The surface $\mathscr{S}_n$ is embedded and incompressible in $M_n \setminus C_n$; thus, every neighborhood of X_n^R contains a properly embedded surface S_n^R which is properly homotopic to $\mathscr{S}_n$ in $M_n \setminus C_n$ [FHS]. We remark that the surfaces S_n^R and $\mathscr{S}_n$ bound a trivial interval bundle in M_n. As in the previous cases, we consider the surfaces $\phi_n^{-1}(S_n^R)$ in N_G and obtain a surface S^R which does not intersect $B_R(\omega_G)$, which is incompressible in $N_G \setminus \pi(C)$ and with $\phi_n(S^R)$ homotopic to S_n^R for an infinite set $\mathscr{I}_R \subset \mathbb{N}$. Each of the surfaces S^R separates N_G and we are going to show that there is R such that one of the components E of $N_G \setminus S^R$ is homeomorphic to $S^R \times \mathbb{R}_+$. Once we have proved this, we obtain as above that the almost isometric embeddings ϕ_n can be extended in the desired way.

In the case that S^R is incompressible in N_G for some R then the same proof as in Case 2 applies. Thus, we assume thus that S^R is compressible for all R. We choose a sequence $R_n \to \infty$ such that S^{R_n} is disjoint from S^{R_m} for all n and m. We claim that, up to choice of a subsequence, the surfaces S^{R_n} and S^{R_m} bound trivial interval bundles for all n and m. As in Canary [Can2] we obtain a collection Γ of disjoint simple closed curves on the boundary $\partial\pi(C)$ of the image of the compact core with the following properties:

1. Γ intersects at least three times every essential simple closed compressible curve on $\partial\pi(C)$,
2. Γ intersects the boundary of every essential and properly embedded annulus $(A, \partial A) \subset (\pi(C), \partial\pi(C))$,
3. $0 = [\Gamma] \in H_1(\pi(C); \mathbb{Z})$ and
4. the collection Γ is freely homotopic in N_G to a collection Γ_* of primitive geodesics.

The collection Γ_* is perhaps not the disjoint union of simple geodesics in N_G but Canary [Can1, Can2] proved that N_G admits a metric g with pinched negative curvature which coincides with the hyperbolic metric of N_G outside of a small neighborhood of Γ_* and such that Γ is homotopic in (N_G, g) to a disjoint union Γ_*^g of simple geodesics in (N_G, g). The collection Γ_*^g is homologically trivial and, therefore, there is an embedded surface $\Sigma \subset N_G$ with $\partial\Sigma = \Gamma_*$. The surface Σ induces a 3-fold cyclic branched cover $\sigma : N_G^3 \to N_G$. Let K be a compact set which contains the surface Σ, the image $\pi(C)$ of the the compact core and the track of a homotopy of Γ_* to Γ, and let K^3 be the pre-image of K under σ. Without loss of generality we assume that none of the surfaces S^{R_n} intersect K; hence S^{R_n} lifts homeomorphically to a surface $\hat{S}^{R_n}$ in $N_G^3 - K^3$ for all n.

The proof of the following Proposition is, word-for-word, the same as the proof of Proposition 16 in [Sou].

Proposition 6.5. *The surfaces $\hat{S}^{R_n}$ are incompressible and represent only finitely many proper homotopy classes.* $\square$

It follows from Waldhausen's co-bordism theorem [Wald] that the surfaces $\hat{S}^{R_n}$ and $\hat{S}^{R_m}$ bound a submanifold of N_G^3 homeomorphic to $\hat{S}^{R_n} \times [0,1]$ for infinitely many, say all, n and m. The branched covering σ is one-to-one on this interval bundle; thus, the surfaces S^{R_n} and S^{R_m} bound an interval bundle in N_G. This yields the desired tame end E of N_G and concludes the proof of Proposition 6.1.

7 Essential surfaces in the geometric limit

In this section we apply Proposition 6.1 fill in the proof of the remaining missing ingredient in the proof of Theorem 1.1. We again remind the reader that we assume the *working assumption* of section 4 holds.

Proposition 7.1. *Assume the geometric limit N_G is not a product. Then there is a properly embedded finite type surface $S \hookrightarrow N_G$ so that one component E of $N_G \setminus S$ is homeomorphic to $S \times \mathbb{R}_+$ and the surface S either is compressible or contains an essential simple closed curve γ that is homotopic into a cusp in N_G but not entirely within E.*

Proof. To begin the proof, we assume that we have the output of Proposition 6.1. Let $N_C = N_G \setminus E$, and let N_S denote the cover of N_G corresponding to $\pi_1(S)$. We will denote again by E the isometric lift of E to N_S, and we let $P_{\partial S} \subset P_S$ denote the subset of the cuspidal part P_S of N_S corresponding cusps to which S is asymptotic.

If N_S is doubly degenerate, then by an application of the covering theorem (see [Th1, Can3]) the geometric limit N_G is also doubly degenerate, and is therefore a product.

Note that we may assume each relative end of N_G that lies in E is degenerate, since otherwise N_G has non-empty conformal boundary and we can conclude that M is tame by [BBES].

Thus, if N_S is not doubly degenerate we claim we are in one of two situations.

Claim. *If N_S is not doubly degenerate then either*

1. *S^0 is compressible and there is an essential non-peripheral simple closed curve $\gamma \subset S^0$ that is homotopically trivial in N_S, or*
2. *the surface S^0 is incompressible and there is an essential, non-peripheral simple closed curve γ homotopic in $N_S \setminus E$ into a cusp P of N_S.*

Noting that condition two guarantees that the projection of γ to N_G is homotopic into a cusp of N_G but not entirely within the image of E, it suffices to prove this claim to verify Proposition 7.1.

Evidently, the only alternative to these options is that S is incompressible and the end of $N_S \setminus P_{\partial S}$ that maps to N_C is geometrically finite with no extra cusps. Given $\varepsilon \in (0,1)$ let $\mathscr{S}_\varepsilon$ denote the strictly convex boundary of the ε-neighborhood of the convex core of N_S, and let $\mathscr{S}_\varepsilon^0$ denote $\mathscr{S}_\varepsilon \cap (N_S \setminus \mathrm{int}(P_S))$. We note that the assumption that the end of N_S mapping to N_C is geometrically finite with no extra cusps guarantees that the surface $\mathscr{S}_\varepsilon$ is connected.

Let B denote the metric completion of the connected component of $N_S \setminus (S \cup \mathscr{S}_\varepsilon)$ containing S and $\mathscr{S}_\varepsilon$ in its closure, and let B^0 denote its compact intersection with $N_S \setminus \text{int}(P_S)$. Then we may choose a compact subset K about the base-frame in N_S that properly contains B^0 and note that the almost-isometries

$$\phi_n \colon N_G \dashrightarrow M_n$$

lift to local almost isometries that converge to a local isometry on K.

By the assumption that the convergence $M_n \to N_G$ is geometrically type-preserving, we may moreover choose ϕ_n to be cusp preserving: ϕ_n sends $K \cap \partial P_G$ to ∂P_n where P_n is the parabolic locus of M_n. Applying Proposition 6.1 we may extend each ϕ_n to a mapping ϕ_n^E on E so that $\phi_n^E|_E$ is a proper embedding.

Since we will work almost entirely with the cover N_S, we denote the lifts of the almost isometries to the appropriate subsets of N_S again by ϕ_n and their extensions by ϕ_n^E. Since the end E lifts to N_S, it follows that for n sufficiently large, say for all n, the mappings ϕ_n^E are local homeomorphisms on the union $K \cup E$ that restrict to embeddings on E.

Since the surfaces $\phi_n(S)$ satisfy the conclusions of Proposition 6.1, we may assume after passing to a subsequence that there are essential non-peripheral simple closed curves γ_n on S so that either

1. $\phi_n(\gamma_n)$ is compressible, or
2. $\phi_n(\gamma_n)$ is homotopic into a cusp of M_n, but not entirely within $\phi_n^E(E)$.

We will work with a single n whose value we will increase as necessary.

The compressible case. Assume first that the simple closed curve $\eta = \phi_n(\gamma_n)$ on $\phi_n(S)$ is compressible. Then we let D be a ruled compressing disk for η, obtained by joining a fixed basepoint to each point along the curve η by geodesics: letting g_t be the geodesic joining $\eta(0)$ to $\eta(t)$ homotopic to $\eta([0,t])$ rel endpoints, we obtain a homotopy of η to the identity by taking $D(t)$ to be the sub-disk of D bounded by the geodesic g_t and the sub-arc $\eta([t,1])$.

Assume for the moment that $P_{\partial S}$ is empty (i.e. S is closed). Since ϕ_n is a covering map with a well defined inverse ϕ_n^{-1} on a neighborhood of η, we may extend ϕ_n^{-1} over D_t for a maximal closed interval of parameter values $t \in [a,1] \subset [0,1]$. If $a > 0$, then there is a value $b \in [a,1]$ for which $\phi_n^{-1}(g_b)$ first touches the strictly convex closed surface $\mathscr{S}_\varepsilon \subset K$. But ϕ_n^{-1} is very close to an isometry on any neighborhood U of a point $x \in g_b$ for which $\phi_n^{-1}(x)$ lies in B, so we may assume the geodesic curvature of $\phi_n^{-1}(g_b)$ is as close to zero as we like for any portion of its length that lies within B. In a neighborhood of the intersection of $\phi_n^{-1}(g_b)$ with $\mathscr{S}_\varepsilon$, then, is a nearly geodesic segment σ, tangent to a strictly convex surface with both endpoints lying within the convex set bounded by $\mathscr{S}_\varepsilon$, a contradiction (see Figure 1).

The only alternative is that $a = 0$, and therefore that ϕ_n^{-1} may be extended over the entirety of D. This contradicts that γ_n is homotopically essential in S.

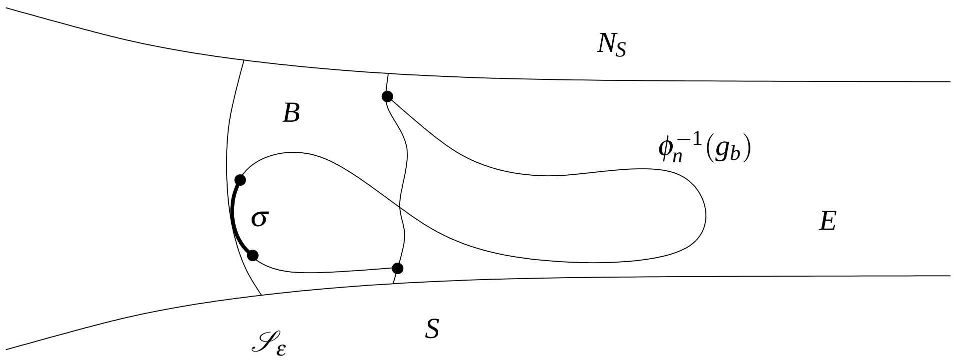

Figure 1. The segment σ must bend to remain within $\mathscr{S}_\varepsilon$.

When $P_{\partial S}$ is non-empty, we modify the argument as follows. We may assume that γ_n has been chosen to avoid $P_{\partial S}$, and it follows that η avoids the corresponding cusps of M_n. If a geodesic g_t intersects a cusp of $P_{\partial S}$, then we may project each arc of intersection to $\partial P_{\partial S}$ along geodesics orthogonal to $\partial P_{\partial S}$. If we let p_t denote the path obtained via this process, then p_t is a broken path consisting of geodesics in $N_G \setminus P_{\partial S}$ together with horocyclic arcs, which are geodesics in the flat metric on $\partial P_{\partial S}$. Let D' denote the compression of η obtained using the paths p_t rather than the geodesics g_t.

For each point $x \in p_t$ for which $\phi_n^{-1}(x)$ lies in $P_{\partial S} \cap B$, geodesics through x in the flat metric on ∂P_n are mapped to arcs in $\partial P_{\partial S}$ with small geodesic curvature at $\phi_n^{-1}(x)$ in the flat metric on $\partial P_{\partial S}$. Taking cusps determined by a smaller $\varepsilon_0 < \mu$ if necessary, we may assume that the convex core boundary surface X intersects $P_{\partial S}$ orthogonally in a finitely bent, convex, piecewise geodesic in the flat metric on $\partial P_{\partial S}$ (see, e.g., [Th1, Ch. 8]). Hence, the intersection $\mathscr{S}_\varepsilon \cap P_{\partial S}$ is convex in the flat metric on $\partial P_{\partial S}$, and the same argument as above applies to show that ϕ_n^{-1} can be extended over all of D', contradicting the incompressibility of γ_n.

The incompressible case. In the case that η is homotopic into a cusp, a similar argument shows that we may construct a homotopy A of η to this cusp consisting of geodesically ruled disks or disks made up of paths that are alternating sequences of geodesics and horocycles.

Precisely, choosing a basepoint y_0 on η, we may parameterize η by $\eta\colon [0,1] \to M_n$ so that $\eta(0) = \eta(1) = y_0$. Joining y_0 to each point $\eta(t)$ by a geodesic g_t homotopic rel endpoints to $\eta([0,t])$ we again obtain a ruled disk D_0 giving a homotopy of η to the based geodesic η_0 at y_0 homotopic to η. We may then choose another point $y_1 \neq y_0$ on η_0 and let D_1 be the ruled disk describing a similar homotopy from η_0 to the based geodesic η_1 at y_1 homotopic to η. Continuing inductively, we form a homotopy A of η to its cusp $\mathbb{P}_\eta$ in P_n so that A is made up of such ruled disks. As before, when any geodesic in a ruling of a disk D_i intersects a cusp of P_n other than $\mathbb{P}_\eta$, we may project this arc of intersection to a horocycle in ∂P_n, and likewise we may always choose y_i to lie outside of $P_n \setminus \mathbb{P}_\eta$.

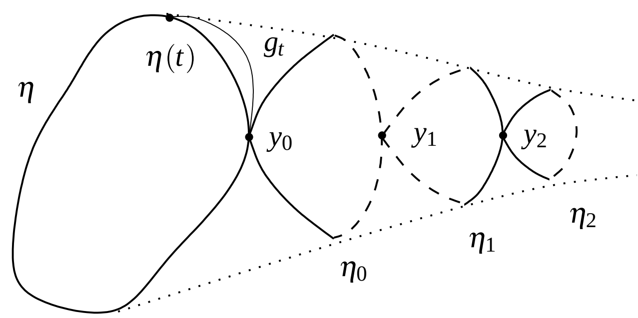

Figure 2. A piecewise ruled homotopy of η to a cusp.

As in the above argument, we may extend ϕ_n^{-1} over all of A since convexity of $\mathscr{S}_\varepsilon$ again prevents the pre-image of any of the geodesics that make up A from intersecting $\mathscr{S}_\varepsilon$ in its interior (and y_i lies in the interior of η_{i-1}, for all $i \geq 1$). Since A contains arbitrarily short representatives of the homotopy class of η, it follows that A gives a homotopy of η into a cusp of M_n. Moreover, this cusp must lie within the image ϕ_n^E since A lies in the complement of $\mathscr{S}_\varepsilon$ and η is non-peripheral in S.

Since E is a product, it follows that η is homotopic entirely within $\phi_n^E(E)$ to a cusp, which contradicts Proposition 6.1. This contradiction verifies the Claim, and therefore proves the proposition. $\square$

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