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Iteration of mapping classes and limits of hyperbolic 3-manifolds

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Abstract Let $\varphi \in \mathrm{Mod}(S)$ be an element of the mapping class group of a surface S. We classify algebraic and geometric limits of sequences $\{Q(\varphi^i X, Y)\}_{i=1}^{\infty}$ of quasi-Fuchsian hyperbolic 3-manifolds ranging in a Bers slice. When φ has infinite order with finite-order restrictions, there is an essential subsurface $D_\varphi \subset S$ so that the geometric limits have homeomorphism type $S \times \mathbb{R} - D_\varphi \times \{0\}$. Typically, φ has pseudo-Anosov restrictions, and D_φ has components with negative Euler characteristic; these components correspond to new asymptotically periodic simply degenerate ends of the geometric limit. We show there is an $s \geq 1$ depending on φ and bounded in terms of S so that $\{Q(\varphi^{si} X, Y)\}_{i=1}^{\infty}$ converges algebraically and geometrically, and we give explicit quasi-isometric models for the limits.

Contents

1 Introduction

The goal of developing a complete understanding of hyperbolic structures on 3-manifolds has given rise to a powerful deformation theory. This deformation theory

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has an algebraic nature, as it parametrizes hyperbolic 3-manifolds with a given fundamental group.

The deformation theory is geometrically quite coarse: important geometric information can be lost in the passage to limits. Indeed, consider two familiar limiting processes: given an essential simple closed curve δ on the boundary of a hyperbolic 3-manifold M, pinching δ and Dehn twisting about δ describe two routes to the *same* algebraic limit manifold M_∞ on the boundary of the deformation space. But while the pinched manifolds geometrically resemble M_∞ late in their approach, the geometry of the twisted manifolds converges to that of a hyperbolic 3-manifold N not even homeomorphic to M_∞. To recover this geometric information one considers the *geometric limits* these approaches produce, giving rise to the study of *algebraic and geometric limits* of hyperbolic 3-manifolds.

One might envision a classification of algebraic and geometric limits. In this paper, we take an initial step in this direction by considering algebraic and geometric limits that arise from a minimal amount of data: a pair of homeomorphic finite-area hyperbolic Riemann surfaces X and Y and a *mapping class* φ. To state our results we review the basic setting.

Let S be an oriented surface of negative Euler characteristic (assume S is closed for simplicity). Let $\text{Teich}(S)$ denote its *Teichmüller space* with its automorphism group $\text{Mod}(S)$, the *mapping class group*. Let $\Gamma(X, Y)$ denote the quasi-Fuchsian *Bers simultaneous uniformization* of $(X, Y) \in \text{Teich}(S) \times \text{Teich}(\overline{S})$; then $\Gamma(X, Y)$ determines $Q(X, Y) = \mathbb{H}^3/\Gamma(X, Y)$ as its quotient hyperbolic 3-manifold.

The quasi-Fuchsian manifolds $QF(S)$ lie in the subspace $AH(S)$ of the *representation variety* $\mathcal{V}(\pi_1(S)) = \text{Hom}(\pi_1(S), \text{PSL}_2(\mathbb{C}))/\text{conj}$ consisting of of faithful representations with discrete (Kleinian) image ($\mathcal{V}(\pi_1(S)$ carries the *algebraic topology*). Each conjugacy class $[\rho] \in AH(S)$ determines a hyperbolic 3-manifold $M = \mathbb{H}^3/\rho(\pi_1(S))$.

One obtains a *Bers slice* $B_Y = \{Q(X, Y) \mid X \in \text{Teich}(S)\}$ in $QF(S)$ by fixing Y in the second factor: B_Y is a precompact copy of Teichmüller space in $AH(S)$. Its closure $\overline{B_Y}$ gives a *Bers compactification* of Teichmüller space, and a resulting *Bers boundary* ∂B_Y. A mapping class $\varphi \in \text{Mod}(S)$ naturally determines a sequence in a Bers slice via its *iteration* $\{Q(\varphi^i X, Y)\}_{i=1}^{\infty} \subset B_Y$ on $X \in \text{Teich}(S)$.

Geometric convergence refers to convergence in the *Hausdorff topology* of the images $\{\rho_i(\pi_1(S))\} = \Gamma_i$ after conjugating so $\rho_i \to \rho$: i.e. $\{\Gamma_i\}$ converges to Γ if

1. For any $\gamma \in \Gamma$ there are $\gamma_i \in \Gamma_i$ so that $\gamma_i \to \gamma$, *and*
2. if elements $\gamma_{i_j} \in \Gamma_{i_j}$ converge, then their limit lies in Γ.

Then $N = \mathbb{H}^3/\Gamma$ is the *geometric limit*. Any convergent sequence in $AH(S)$ has a geometric limit after passing to a subsequence. Our main result is (theorem 7.3):

Theorem 1 HOMEOMORPHISM TYPES. *There is an essential subsurface $D_\varphi \subset S$ naturally associated to φ so that any geometric limit N of $\{Q(\varphi^i X, Y)\}_{i=1}^{\infty}$ has the homeomorphism type*

$$N \cong S \times \mathbb{R} - D_\varphi \times \{0\}$$

if $D_\varphi \neq S$ and $N \cong S \times \mathbb{R}$ otherwise.

The subsurface $D_\varphi \subset S$ records the locus of the infinite order dynamics of φ: the complement $S - D_\varphi$ is the maximal subsurface on which φ restricts to a finite order mapping class (see definition 2.10). The limit representation ρ has image $\rho(\pi_1(S)) < \Gamma$, so the algebraic limit $Q_\infty = \mathbb{H}^3/\rho(\pi_1(S))$ covers N by a local isometry.

One may ask whether the passage to subsequences is necessary. In particular, the question of *convergence* of iteration of mapping classes on a Bers slice was originally posed by L. Bers and answered by J. Cannon and W. Thurston the case when φ is pseudo-Anosov, i.e. no power of φ stabilizes any non-peripheral essential simple closed curve up to isotopy.

As each algebraic accumulation point $Q_\infty \in \overline{B_Y}$ and each geometric accumulation point N are hyperbolic 3-manifolds, one approach to convergence is to compare accumulation points geometrically. We go on to answer Bers question in general by giving explicit models for the algebraic and geometric limits of iteration. Much of the work lies in the following:

Theorem 2 QUASI-ISOMETRY TYPES. *Let $\{Q(\varphi^i X, Y)\}_{i=1}^\infty$ have geometric limit N. Then the quasi-isometry type of N depends only on the mapping class φ.*

A more precise formulation appears in proposition 7.1 and theorem 7.2 (§7).

Iteration of finite-order mapping classes does not converge; one must first pass to a finite power to obtain a convergent sequence. In general, there is an integer s so that passing to a power φ^s that is *stable* (each finite order restriction of φ^s is the identity) ensures convergence. In other words, we have (theorem 6.1):

Theorem 3 ITERATION CONVERGES. *Let $\varphi \in \mathrm{Mod}(S)$ be a mapping class. Then there is an $s \geq 1$ depending on φ and bounded in terms of S so that the sequence $\{Q(\varphi^{si} X, Y)\}_{i=1}^\infty$ converges algebraically and geometrically.*

How φ determines the quasi-isometry type of N is revealed over the course of the paper. We illustrate the process in key examples, given below.

Examples

In analyzing any sequence $\{Q(X_i, Y)\}_{i=1}^\infty$ in a Bers compactification $\overline{B_Y}$ natural questions arise. What happens to the surfaces X_i in the limit? Which elements have become parabolic? Which ends of the algebraic limit are geometrically finite, and which degenerate? What is the geometric limit? Here are the answers for three basic examples of mapping class iteration.

I. $\psi \in \mathrm{Mod}(S)$ is pseudo-Anosov. A mapping class $\psi \in \mathrm{Mod}(S)$ is *pseudo-Anosov* if no power of ψ preserves the isotopy class of any essential simple closed curve on S. For this case, the surfaces $\psi^i X$ degenerate, leaving a *totally degenerate* limit Q_ψ (i.e. $\partial Q_\psi = Y$) with no new parabolic elements. The algebraic and geometric limits of $Q(\psi^i X, Y)$ *agree* giving an example of *strong convergence*.

By contrast, the following iterations do *not* converge strongly.

II. $\vartheta \in \mathrm{Mod}(S)$ is a Dehn twist. Under iteration of a *Dehn twist* ϑ about δ (figure 1) the algebraic and geometric limits of $Q(\vartheta^i X, Y)$ differ. In the algebraic limit,

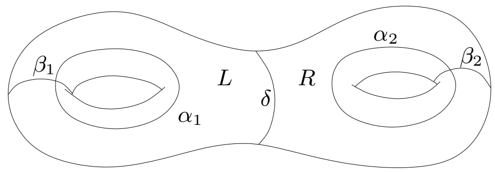


Figure 1. Examples of mapping classes.

the surfaces $\vartheta^i X$ have split into two quasi-Fuchsian punctured tori corresponding to L and R. The induced representations ρ_i converge (up to conjugacy) on δ to a *parabolic* element, while the cyclic groups $\langle \rho_i(\delta) \rangle$ converge geometrically to a *rank*-2 parabolic group (isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$), indicating the presence of a new *torus-end* $E \cong T^2 \times [0, \infty)$ in the geometric limit N; N has homeomorphism type

$$N \cong S \times \mathbb{R} - \delta \times \{0\}.$$

Our investigation is motivated by the following example.

III. $\varphi \in \mathbf{Mod}(S)$ is *partially* pseudo-Anosov. Let $\varphi = \vartheta_{\alpha_2} \circ \vartheta_{\beta_2}$ the product of Dehn twists about α_2 and β_2 (figure 1). The induced mapping class $\varphi|_R \in \mathrm{Mod}(R)$ is pseudo-Anosov; φ is called the *half-pseudo-Anosov* mapping class.

In the algebraic limit Q_φ, the Riemann surfaces $\varphi^i X$ *partially degenerate* along the subsurface R: the curve δ is parabolic and $Q_\varphi - \{\text{cusps}\}$ has a quasi-Fuchsian end $L \times \mathbb{R}^+$ as well as a *degenerate end* $R \times \mathbb{R}^+$.

The geometric limit N has the homeomorphism type

$$N \cong S \times \mathbb{R} - R \times \{0\}.$$

Now the *subsurface* R recedes to infinity, leaving new degenerate ends E_R and E'_R of $N - \{\text{cusps}\}$ in its wake; each is asymptotically periodic by $\varphi|_R$ (see figure 2).

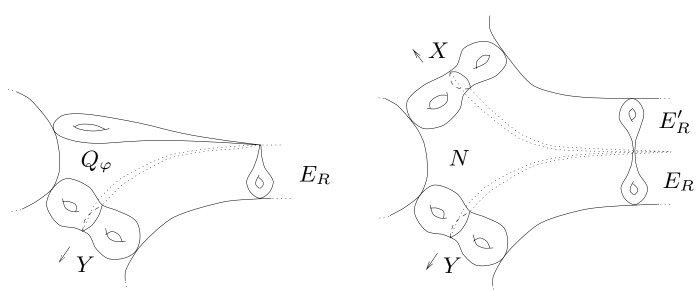


Figure 2. The algebraic and geometric limits of $\{Q(\varphi^i X, Y)\}_{i=1}^{\infty}$.

Outline of the proof

We outline the proof that iteration of the half-pseudo-Anosov class φ (example III, above) converges and describe the quasi-isometry type of its geometric limit.

Step 1) Quasi-isometry invariants. Thurston's theory of pleated surfaces reveals that in any algebraic accumulation point Q_φ of $\{Q(\varphi^i X, Y)\}_{i=1}^\infty$ in ∂B_Y, δ has become parabolic, and $Q_\varphi - \{\text{cusps}\}$ has developed a simply degenerate end $E_R \cong R \times \mathbb{R}^+$. The end E_R has the property that for any essential simple closed curve $\gamma \subset R$, the curves $\varphi^i(\gamma) \subset R$ determine closed geodesics $\varphi^i(\gamma)^*$ in Q_φ that leave every compact subset of Q_φ.

Step 2) Asymptotic geometry. Actually, the end E_R is *asymptotically periodic*: E_R is quasi-isometric to one end of the periodic $\mathbb{Z}$-cover $M_{\varphi^{-1}|_R}$ of the mapping torus

$$T_{\varphi^{-1}|_R} = R \times I/(x,0) \sim (\varphi^{-1}|_R(x), 1)$$

with quasi-isometry constant tending to 1 out the end.

It follows that the quasi-isometric geometry of E_R depends only on the isotopy class $\varphi|_R$. Since any two algebraic accumulation points Q_φ and Q'_φ have their topology, parabolic locus, and corresponding ends all determined by φ up to quasi-isometry, the quasi-isometry type of any algebraic limit depends only on φ. Hence, any two algebraic accumulation points Q_φ and Q'_φ admit a quasi-isometry compatible with markings.

Step 3) The geometric limit. By a *re-marking trick*, any geometric limit N is also the geometric limit of a subsequence of $\varphi^{-i}(Q(\varphi^i X, Y)) = Q(X, \varphi^{-i} Y)$: the same sequence of manifolds *re-marked* by φ^{-i}. Thus, N is also covered the algebraic limit $Q_{\varphi^{-1}}$ of this re-marked sequence, which has a similar but inverted structure.

By a *gluing lemma* (lemma 6.5) the covers Q_φ and $Q_{\varphi^{-1}}$ are compatible with *gluing data* τ: an orientation-reversing involution of the *quasi-Fuchsian boundary* $\partial_{\text{qf}}(Q_\varphi \sqcup Q_{\varphi^{-1}})$ (the subset of the conformal boundary of $Q_\varphi \sqcup Q_{\varphi^{-1}}$ whose corresponding covers are quasi-Fuchsian). The limits Q_φ and $Q_{\varphi^{-1}}$ may be glued (by Klein-Maskit combination) along quasi-Fuchsian ends corresponding to L to form a complete hyperbolic manifold $(Q_\varphi \sqcup Q_{\varphi^{-1}})/\tau$ that also covers N by a local isometry. The covering extends to an embedding on the conformal boundary $\partial\left((Q_\varphi \sqcup Q_{\varphi^{-1}})/\tau\right) = X \sqcup Y$ into ∂N, so the cover is an isometry. Because the gluing τ identifies the quasi-Fuchsian ends of Q_φ and $Q_{\varphi^{-1}}$ corresponding to L, the geometric limit N has homeomorphism type $N \cong S \times \mathbb{R} - R \times \{0\}$ (cf. theorem 1).

Step 4) Geometric convergence. Any two geometric accumulation points N and N' are realized as gluings $(Q_\varphi \sqcup Q_{\varphi^{-1}})/\tau$ and $(Q'_\varphi \sqcup Q'_{\varphi^{-1}})/\tau$ of pairs of algebraic accumulation points. Thus, Step 2) implies there is a quasi-isometry $\Theta\colon N \to N'$ whose lifts to Q_φ and $Q_{\varphi^{-1}}$ are compatible with markings. Since $\partial N = X \sqcup Y = \partial N'$, Θ is homotopic to an isometry ξ, so the sequence converges geometrically.

Step 5) Algebraic convergence. The isometry ξ lifts to a marking-preserving isometry $\widetilde{\xi}\colon Q_\varphi \to Q'_\varphi$, so the sequence converges algebraically.

Remarks on the general case: Examples I, II, and III present the prototypes for iteration of general mapping classes on B_Y; the arguments fit together to handle the general case, after passing to an iterate of φ whose finite order behavior stabilizes.

Though our proof makes use of Thurston's construction of a hyperbolic structure on $T_{\varphi|_R}$, a quasi-isometrically correct model for the end E arising in Q_φ can be given *combinatorially* by gluing a half-infinite collection of copies of $R \times I$ together end-to-end by $\varphi|_R$ (the analogous construction applies to $Q_{\varphi^{-1}}$). Hence, a concrete quasi-isometric model for the gluing $(Q_\varphi \sqcup Q_{\varphi^{-1}})/\tau$, and hence for the geometric limit, may be constructed directly. We discuss this in section 7.

Limit Sets

The action of the Kleinian covering group Γ_i for $Q(\varphi^i X, Y)$ partitions $\widehat{\mathbb{C}}$ into its *limit set* Λ_i, where it acts chaotically, and its domain of discontinuity $\Omega_i = \widehat{\mathbb{C}} - \Lambda_i$. While Λ_i converges in the Hausdorff topology to the limit set Λ_G of the geometric limit Γ_G of Γ_i, the limit set Λ_A of the algebraic limit is strictly smaller than Λ_G when the convergence is not strong. That the component $\Omega_Y \subset (\widehat{\mathbb{C}} - \Lambda_A)$ covering $Y \subset \partial Q_\varphi$ *embeds* in $\Omega_G = \widehat{\mathbb{C}} - \Lambda_G$ will be an important tool in our proof.

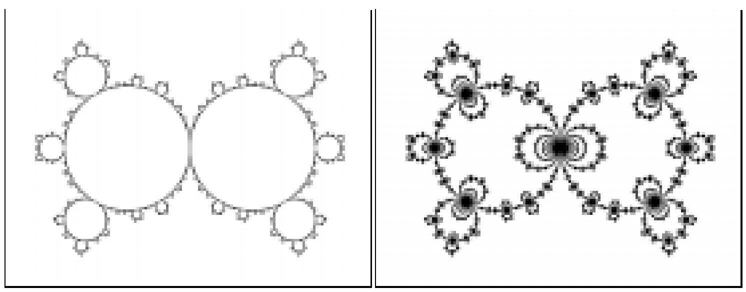


Figure 3. Limit sets for algebraic and geometric limits for example II.

In figures 3 and 4 we have rendered[1] the limit sets of the algebraic and geometric limits in examples II and III for certain X and Y. In figure 3, Ω_Y contains the point at infinity and embeds in $\widehat{\mathbb{C}} - \Lambda_G$. In figure 4, Ω_Y is the central component with non-circle boundary; it embeds into a portion of the upper hemisphere in $\widehat{\mathbb{C}} - \Lambda_G$ after a $\mathrm{PSL}_2(\mathbb{C})$ change of coordinates.

History and References

The question of convergence of iteration was originally raised in [Bers3] wherein L. Bers proves that any accumulation point of pseudo-Anosov iteration is totally degenerate and free of accidental parabolics.

[1] We employ computer programs of Curt McMullen.

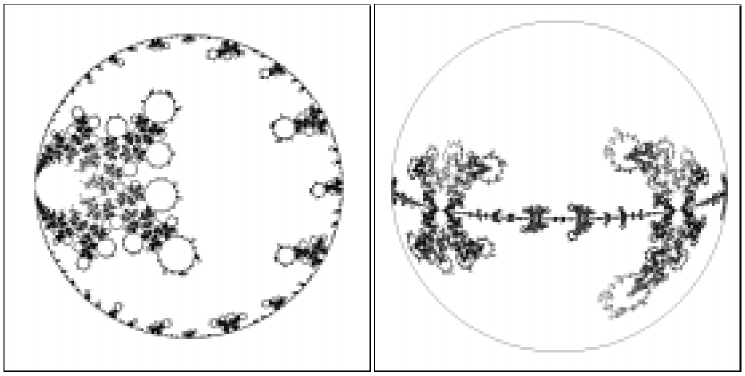


Figure 4. Limit sets for algebraic and geometric limits for example III.

Algebraic and geometric convergence for the pseudo-Anosov case were later proven in [CT, §7]; a detailed proof has appeared more recently in [Mc2, §3]. An expository account of examples II and III appears in [Br1]. The geometric limits in examples I and II are employed: (I) in Thurston's geometrization of 3-manifolds fibering over the circle [Th4] [Otal1] [Mc2], and (II) in the proof of non-continuity of the action of the modular group on $\overline{B_Y}$ [KT]. Our study completes the general picture of algebraic and geometric limits that arise under iteration of $\varphi \in \mathrm{Mod}(S)$ on a Bers slice.

We remark that the homeomorphism (and thence quasi-isometry) type of the geometric limit of half-pseudo-Anosov iteration differs from other examples of non-strong convergence presented in [KT, §3], [Th4, §7], and [BO] in which all new ends of the geometric limit are rank-2 cusps.

Plan of the paper: Section 2 presents necessary background, and section 3 introduces results from pleated surface theory. Section 4 builds up a complete picture of the quasi-isometry invariants of any algebraic accumulation point Q_φ. Section 5 applies Thurston's construction of hyperbolic structures on 3-manifolds fibering over the circle to prove that the ends arising from induced pseudo-Anosov dynamics of φ are asymptotically periodic; this determines the quasi-isometry type of Q_φ.

In Section 6 we prove convergence of iteration, giving a direct construction of the geometric limit by gluing together algebraic limits. Finally, section 7 describes quasi-isometric models for algebraic and geometric limits implicit in section 6. We show how these models can be constructed concretely from φ without reference to specific hyperbolic structures, thereby elucidating how the homeomorphism type changes in the geometric limit.

Acknowledgments. This paper presents results of my doctoral dissertation [Br3] completed at U.C. Berkeley. I would like to thank my advisor, Curt McMullen, for his suggestions and guidance, and the referees for many useful comments.

2 Preliminaries

Surfaces. Let S be a compact connected oriented topological surface of negative Euler characteristic. When S has non-empty boundary, denote by $\mathrm{int}(S)$ its interior $S - \partial S$.

The *Teichmüller space* $\mathrm{Teich}(S)$ of S parametrizes finite area hyperbolic structures on S up to isotopy: finite area hyperbolic surfaces X, each equipped with an orientation-preserving homeomorphism, or *marking* $f\colon \mathrm{int}(S) \to X$ with the equivalence

$$(f, X) \sim (g, Y)$$

when there is an orientation-preserving isometry $\phi\colon X \to Y$ such that $\phi \circ f$ is homotopic to g.

An element $\beta \in \pi_1(S)$ is *peripheral* if it is freely homotopic to a component of ∂S. Any element $(f, X) \in \mathrm{Teich}(S)$ determines a discrete faithful representation $f_*\colon \pi_1(S) \to \mathrm{PSL}_2(\mathbb{R}) = \mathrm{Isom}^+(\mathbb{H}^2)$ up to conjugacy, such that f_* sends peripheral elements of $\pi_1(S)$ to parabolic elements of $\mathrm{Isom}^+(\mathbb{H}^2)$.

A *subsurface* $R \subset S$ of a compact oriented surface S is a compact 2-dimensional submanifold of S. An *essential subsurface* is a subsurface $R \subset S$ whose boundary is homotopically essential. When $R = R_1 \sqcup R_2$ is a disjoint union of compact oriented surfaces, each with negative Euler characteristic, we define $\mathrm{Teich}(R) = \mathrm{Teich}(R_1) \times \mathrm{Teich}(R_2)$. We will often refer to surfaces $X \in \mathrm{Teich}(S)$ suppressing the implicit marking.

Kleinian groups and hyperbolic 3-manifolds. A *Kleinian group* Γ is a discrete subgroup of $\mathrm{Isom}^+\mathbb{H}^3 = \mathrm{PSL}_2(\mathbb{C})$. Naturally associated to any Kleinian group Γ are its *limit set* Λ where any orbit $\Gamma(x)$, $x \in \mathbb{H}^3$, accumulates on $\widehat{\mathbb{C}}$, its *domain of discontinuity* $\Omega = \widehat{\mathbb{C}} - \Lambda$ where the action of Γ is properly discontinuous, and the *convex hull* $\mathrm{ch}(\Lambda)$ of Λ, the smallest hyperbolically convex subset in $\mathbb{H}^3$ whose closure in $\mathbb{H}^3 \cup \widehat{\mathbb{C}}$ contains the limit set. When necessary, we will use the notation $\Omega(\Gamma)$, $\Lambda(\Gamma)$ to denote the domain of discontinuity and limit set of Γ.

When Γ is torsion-free, the quotient $M = \mathbb{H}^3/\Gamma$ is a complete hyperbolic 3-manifold. By adjoining the domain of discontinuity to $\mathbb{H}^3$ and passing to the quotient, we extend M to its *Kleinian manifold*

$$\overline{M} = \left(\mathbb{H}^3 \cup \Omega\right)/\Gamma$$

with its *conformal boundary* $\partial M = \Omega/\Gamma$. The quotient $\mathrm{ch}(\Lambda)/\Gamma = \mathrm{core}(M)$ is the minimal convex subset of M called its *convex core*. If a metric ϵ-neighborhood $\mathcal{N}_\epsilon(\mathrm{core}(M))$ of the convex core of M has finite volume, then M and its Kleinian uniformization Γ are *geometrically finite*. Otherwise, they are *geometrically infinite*.

Surface groups. Let $H(S)$ be the set of all complete hyperbolic 3-manifolds M equipped with homotopy equivalences $f\colon S \to M$ such that f_* sends peripheral elements to parabolics, with the equivalence $(f_1\colon S \to M_1) \sim (f_2\colon S \to M_2)$ when there is an orientation-preserving isometry $\phi\colon M_1 \to M_2$ such that $\phi \circ f_1$ is homotopic to f_2.

A choice of *baseframe* $\omega \in M$ determines a Kleinian group Γ by the requirement that the standard frame $\widetilde{\omega}$ at the origin in $\mathbb{H}^3$ lie over ω in the covering projection

$$(\mathbb{H}^3, \widetilde{\omega}) \to (\mathbb{H}^3, \widetilde{\omega})/\Gamma = (M, \omega).$$

The homotopy equivalence $f\colon S \to (M, \omega)$ from S to the *based* hyperbolic manifold (M, ω) determines a representation $f_*\colon \pi_1(S) \to \mathrm{PSL}_2(\mathbb{C})$. The image is discrete, and f_* is faithful.

Equipping $M \in H(S)$ with the additional data of a baseframe, we obtain the set $H_\omega(S)$ of *marked based hyperbolic manifolds* $(f\colon S \to (M, \omega))$ by requiring ϕ to preserve baseframes. Giving $H_\omega(S)$ the compact-open topology on the representations f_* (the natural topology on $\mathcal{V}(\pi_1(S))$), we obtain the space $AH_\omega(S)$. We give $H(S)$ the quotient topology from the natural map $H_\omega(S) \to H(S)$ obtained by forgetting the baseframe and call the resulting space $AH(S)$.

As with Teichmüller space, a hyperbolic manifold $M \in AH(S)$ and any lift (M, ω) of M to $AH_\omega(S)$ are implicitly marked.

Quasi-isometries. Let M and N be Riemannian n-manifolds. A diffeomorphism $h\colon M \to N$ is called an *L-quasi-isometry*[2] if there is a real number $L > 1$ such that given any non-zero tangent vector $v \in TM$,

$$\frac{1}{L} \le \frac{|Dh(v)|}{|v|} \le L.$$

A quasi-isometry $h\colon M \to N$ has a *quasi-isometry constant* $L(h)$ which is the infimum over all L such that h is an L-quasi-isometry.

The *quasi-isometric distance* $d(.,.)$ on $H(S) \times H(S)$ is defined as follows: if $M_0 = (f\colon S \to M)$ and $N_0 = (g\colon S \to N)$ in $H(S)$ then we define

$$d(M_0, N_0) = \inf_{\{h \,|\, h \circ f \simeq g\}} \log L(h).$$

If there is no orientation-preserving h in the appropriate homotopy class, define $d(M_0, N_0) = \infty$. Let $QH(S)$ denote $H(S)$ with the *quasi-isometric topology* induced by $d(.,.)$. Compactness theorems for quasi-conformal mappings [LV] [Gard, §1.8, Lem. 6] guarantee that $d(.,.)$ is lower semi-continuous [Mc2, Prop. 3.1].

Algebraic and geometric convergence. Let X be a separable metric space. Let $\mathrm{Cl}(X)$ denote its set of closed subsets.

Definition 2.1 *In the* Hausdorff topology *on* $\mathrm{Cl}(X)$*, a sequence* $\{Y_i\}$ *tends to* Z *if*

1. *For every* $z \in Z$*, there are* $y_i \in Y_i$ *such that* $\lim_{i\to\infty} y_i = z$.
2. *For any subsequence* Y_{i_j}*, and elements* $y_{i_j} \in Y_{i_j}$*, if* $y_{i_j} \to z$ *then* $z \in Z$.

The set $\mathrm{Cl}(X)$ is compact in the Hausdorff topology (see [HY, §2-16]). Giving the discrete subgroups of $\mathrm{PSL}_2(\mathbb{C})$ the Hausdorff topology as closed subsets we obtain the *geometric topology* on Kleinian groups.

[2] Our terminology, somewhat non-standard since the advent of coarse geometry, follows [Mc2].

The *geometric topology* on based hyperbolic 3-manifolds (M, ω) is the geometric topology on their Kleinian covering groups Γ. Let $\mathcal{H}^3$ be the set of all based hyperbolic 3-manifolds (M, ω). In intrinsic terms, a sequence of based hyperbolic manifolds $\{(M_i, \omega_i)\}$ converges to a based hyperbolic manifold (N, ω) in the geometric topology if and only if for any compact submanifold $K \subset N$ containing ω, there are quasi-isometries $h_i \colon K \to M_i$ such that $h(\omega) = \omega_i$ and so that h_i tends to an isometry in the C^∞ topology (see [BP, Thm. E.1.13]).

Given a convergent sequence $M_i \to M$ in $AH(S)$, convergent lifts (M_i, ω_i) to $AH_\omega(S)$ determine convergent representations $\rho_i \to \rho$. If the Kleinian groups $\rho_i(\pi_1(S))$ converge geometrically to Γ, it follows from definition 2.1 (part 2) that $\rho(\pi_1(S)) < \Gamma$, so M naturally covers $N = \mathbb{H}^3/\Gamma$. We shall see that $\rho(\pi_1(S))$ can often be a proper subgroup of Γ [KT, §3] [Th1, §9.1].

We now illustrate a baseframe independent notion of geometric convergence of *marked* hyperbolic manifolds in $AH(S)$.

Definition 2.2 *Let marked hyperbolic 3-manifolds $\{M_n\} \subset AH(S)$ converge algebraically to M_∞ in $AH(S)$. Then $\{M_n\}$* converges geometrically *to a limit N if there are convergent lifts $\{(M_n, \omega_n)\}$ to $AH_\omega(S)$ so that the sequence of based hyperbolic 3-manifolds (M_n, ω_n) converges geometrically to (N, ω).*

While the baseframes are necessary to define this notion of geometric convergence, the geometric limit N does not depend on the choices of baseframes.

Proposition 2.3 ALGEBRAIC COVERS GEOMETRIC. *The marked manifolds M_n converge geometrically to a unique geometric limit N after passing to a subsequence. The limit N is covered by M_∞ by a local isometry.*

Proof. Let $(M_n, \omega_n) \to (M_\infty, \omega_\infty)$ be convergent lifts of M_n to $AH_\omega(S)$. Convergence of (M_n, ω_n) implies that ω_n lies in $M_{(r,R)}$ for $0 < r < R < \infty$. By compactness, we may extract a geometric limit (N, ω) covered by M_∞ after passing to a subsequence [Mc2, Prop. 2.4]. Conjugacies $\Phi_n \in \mathrm{PSL}_2(\mathbb{C})$ between representations induced by any two convergent sequences of lifts to $AH_\omega(S)$ converge to a conjugacy of algebraic and geometric limits, so N is unique. □

Definition 2.4 *A sequence $\{M_n\} \subset AH(S)$ converges* strongly *to a limit M_∞ if it converges both algebraically and geometrically to M_∞.*

We will see many examples of sequences that do *not* converge strongly.

Quasi-Fuchsian groups and manifolds. A *quasi-Fuchsian* group Γ is a Kleinian group that preserves a directed Jordan curve in $\widehat{\mathbb{C}}$. The quotient $\Omega(\Gamma)/\Gamma$ is a pair of Riemann surfaces X and Y. Endowed with an isomorphism $\rho\colon \pi_1(S) \to \Gamma$, the Γ-equivariant conformal structures on the two components Ω_X and Ω_Y of $\Omega(\Gamma)$ determine a pair of points in $\mathrm{Teich}(S) \times \mathrm{Teich}(\overline{S})$ ($\overline{S}$ is S with its orientation reversed), and the conjugacy class $[\rho]$ determines an element of $AH(S)$.

Conversely, in [Bers1] Bers exhibited a homeomorphism

$$Q\colon \mathrm{Teich}(S) \times \mathrm{Teich}(\overline{S}) \to QF(S) \subset AH(S)$$

from the product of Teichmüller spaces to the quasi-Fuchsian representations via *simultaneous uniformization* of a pair $(X, Y) \in \text{Teich}(S) \times \text{Teich}(\overline{S})$. The surfaces $X = \partial_c Q(X, Y)$ and $Y = \overline{\partial}_c Q(X, Y)$ make up $\partial Q(X, Y)$.

The convex core boundary $\partial(\text{core}(Q(X, Y)))$ inherits the structure of a pair of hyperbolic surfaces from its path metric (see [EM, Thm. 1.12.1]). Each component is isotopic outside of the convex core to one of the conformal boundary components, which we say it "faces." Let $\partial_h(Q(X, Y))$ and $\overline{\partial}_h(Q(X, Y))$ be the components of the convex core boundary facing X and Y respectively.

Bers' boundary. Bers proved that the slice $B_Y = \{Q(X, Y) \mid X \in \text{Teich}(S)\}$ of $QF(S)$ is an embedded copy of Teichmüller space with compact closure in $AH(S)$. The *Bers slice* $B_Y \subset QF(S)$ and the resulting *Bers boundary* $\partial B_Y \subset AH(S)$ lie central to many issues in the deformation theory of hyperbolic 3-manifolds. In [Bers2], Bers classifies elements $(f : S \to M)$ in ∂B_Y:

1. If $f_*(\gamma)$ is parabolic for some non-peripheral element $\gamma \in \pi_1(S)$, then M is a *cusp* and γ is an *accidental parabolic*.
2. If the conformal boundary ∂M is connected, then $\partial M = Y$, and M is *totally degenerate*. The image $f_*(\pi_1(S))$ is a *totally degenerate group*.
3. If $\partial M = Y \sqcup X_1 \sqcup \ldots \sqcup X_q$ and $0 < \sum_{n=1}^{q} \text{area}(X_n) < \text{area}(Y)$, then M is *partially degenerate*, and $f_*(\pi_1(S))$ is a *partially degenerate group*.

Only cases (2) and (3) are mutually exclusive.

Geodesic and measured laminations. A *geodesic lamination* λ on a finite area hyperbolic surface $X \in \text{Teich}(S)$ is a closed subset of X given as a disjoint union of simple, complete geodesics called *leaves* of the lamination (see [Th1, pp. 8.25] [CB, pp. 39]). We give the geodesic laminations $gl(X)$ the *pleating topology*, in which laminations $\{\lambda_i\}$ converge to a lamination λ if any l in λ is the limit of $l_i \in \lambda_i$ (cf. the *Thurston topology* of [CEG, Def. 4.1.10]). Canonical homeomorphisms between any pair of hyperbolic surfaces X and Y in $\text{Teich}(S)$ induce homeomorphisms of $gl(X)$ and $gl(Y)$. This gives a universal *geodesic lamination space* $\mathcal{GL}(S)$; a point $\lambda \in \mathcal{GL}(S)$ determines a geodesic lamination on any $X \in \text{Teich}(S)$.

Let $\mathbb{S}$ be the set of all isotopy classes of essential non-peripheral simple closed curves on S. As any element $\gamma \in \mathbb{S}$ has a unique geodesic representative on any hyperbolic surface, there is a natural inclusion $\mathbb{S} \hookrightarrow \mathcal{GL}(S)$. The *intersection number* $i : \mathbb{S} \times \mathbb{S} \to \mathbb{Z}_{\geq 0}$ counts the minimal number of transverse intersection points of representatives of $(\gamma, \delta) \in \mathbb{S} \times \mathbb{S}$ in their respective isotopy classes.

Let $\iota : \mathbb{R}_+ \times \mathbb{S} \to \mathbb{R}^{\mathbb{S}}$ be the embedding $\langle \iota(t\gamma) \rangle_\alpha = ti(\alpha, \gamma)$. Then the *measured laminations* $\mathcal{ML}(S)$ on S are obtained by taking the closure of the image $\mathcal{ML}(S) = \overline{\iota(\mathbb{R}_+ \times \mathbb{S})}$. Each measured lamination μ determines a *transversely measured geodesic lamination*; the underlying geodesic lamination $|\mu|$ is called the *support* of μ (see [Bon2, §1] [FLP] [EM, 3.5.1]).

The modular group. For our discussion of the Modular group, let R be a compact oriented, but *possibly disconnected* surface, each component of which has negative Euler characteristic. The *modular group* $\text{Mod}(R)$ is the group of isotopy classes of orientation-preserving self-homeomorphisms of R, called *mapping classes*. The group $\mathrm{Mod}(R)$ acts on $\mathcal{S}(R)$, the isotopy classes of non-peripheral essential simple closed curves on R. Thurston extended this action to $\mathcal{ML}(R) = \mathcal{ML}(R_1) \times \ldots \times \mathcal{ML}(R_q)$ where $R = R_1 \sqcup \ldots \sqcup R_q$ to give a classification of mapping classes [Th3], which we now discuss.

A *partition* of R is a family $\Pi \subset \mathcal{S}(R)$ of distinct isotopy classes of disjoint curves. A mapping class φ is said to be *reduced* by Π if $\varphi(\Pi) = \Pi$. If the only invariant partition for φ is the trivial partition, φ is *irreducible*. Thurston proved that an irreducible mapping class either has finite order, or has the following type:

Definition 2.5 *A mapping class $\psi \in \mathrm{Mod}(R)$ is* pseudo-Anosov *if there exist measured laminations μ^s and μ^u in $\mathcal{ML}(R)$ (the* stable *and* unstable *laminations for ψ) and a real number $c > 1$ so that $\psi(\mu^s) = \frac{1}{c}\mu^s$ and $\psi(\mu^u) = c\mu^u$.*

We now discuss the *reducible* case.

A partition Π of S naturally determines a complementary essential subsurface S_Π up to isotopy each component of which has negative Euler characteristic; S_Π is determined by choosing disjoint oriented open annular neighborhoods $\mathcal{N}(\Pi)$ of disjoint curves representing Π and forming the complement $S_\Pi = S - \mathcal{N}(\Pi)$.

An invariant partition Π for $\varphi \in \mathrm{Mod}(S)$ induces a mapping class $\varphi_\Pi \in \mathrm{Mod}(S_\Pi)$ by restriction. When Π decomposes S into disjoint subsurfaces F and G invariant by φ, denote by $\varphi|_F \in \mathrm{Mod}(F)$ and $\varphi|_G \in \mathrm{Mod}(G)$ the mapping classes naturally induced by restriction. The subsurfaces and induced mapping classes are well defined as isotopy classes. In this language, Thurston's classification takes the following form [Th3, Thm. 4] [FLP, Exp. 9] [BLM, Thm. C].

Theorem 2.6 (Thurston) *Any mapping class $\varphi \in \mathrm{Mod}(S)$ determines a partition Π and essential subsurfaces S_F and S_P of S so that the triple $(\Pi, S_\mathsf{F}, S_\mathsf{P})$ is unique and φ-invariant up to isotopy, and so that:*

1. *S_Π decomposes as $S_\Pi = S_\mathsf{F} \sqcup S_\mathsf{P}$,*
2. *$\varphi_\mathsf{F} = \varphi|_{S_\mathsf{F}}$ has finite order and $\varphi_\mathsf{P} = \varphi|_{S_\mathsf{P}}$ is pseudo-Anosov, and*
3. *Π is minimal among all partitions satisfying (1) and (2).*

The triple $(\Pi, S_\mathsf{F}, S_\mathsf{P})$, called the *Nielsen-Thurston* decomposition of S for φ, is well defined up to isotopy (the uniqueness of Π is proven in [BLM, Thm. C]). Its *finite-order part* φ_F acts on its *finite-order subsurface* S_F and its *pseudo-Anosov part* φ_P acts on its *pseudo-Anosov subsurface* S_P. The *minimal reducing partition* Π is empty if and only if φ is irreducible. When necessary, we use $\Pi(\varphi)$, $S_\mathsf{F}(\varphi)$ and $S_\mathsf{P}(\varphi)$ to denote dependence on the mapping class φ.

We will be interested in the following simplifying property for mapping classes.[3]

Definition 2.7 *A mapping class $\varphi \in \mathrm{Mod}(S)$ is* stable *if its finite order part φ_F is the identity element of* $\mathrm{Mod}(S_\mathsf{F}(\varphi))$.

Since φ_F has finite order, $\varphi \in \mathrm{Mod}(S)$ has a power that is stable.

Definition 2.8 *The* stable power *for φ is the least integer $s \in \mathbb{Z}^+$ such that φ^s is stable. Call φ^s the* first stable iterate *of φ.*

[3] Cf. the similar notion of a *pure* mapping class of [Iv, pp. 3].

By Hurwitz' theorem (see [Gr, Thm. 1.7.1]) s is uniformly bounded in terms of S.

Let δ lie in $\Pi(\varphi)$. We say δ is an *isolated Dehn twisting curve* for φ if there is an essential subsurface $S_\delta \subset S$ for which δ is non-peripheral in S_δ, any stable iterate φ^s preserves S_δ up to isotopy, and $\varphi^s|_{S_\delta}$ represents a power of a Dehn-twist about δ in $\mathrm{Mod}(S_\delta)$. Then the following corollary may be readily verified.

Corollary 2.9 *If δ lies in $\Pi(\varphi)$, then either δ is peripheral in $S_{\mathsf{P}}(\varphi)$ or δ is an isolated Dehn twisting curve for φ.* $\square$

The Nielsen-Thurston decomposition for φ encodes the locus of its infinite order dynamics:

Definition 2.10 *Let φ be an element of* $\mathrm{Mod}(S)$. *The* dynamic subsurface $D_\varphi \subset S$ *is the complement $\overline{S - S_{\mathsf{F}}(\varphi)}$ of the finite order subsurface for φ.*

Note that the subsurface D_φ is essential and contains annular components when φ has isolated Dehn twisting curves.

Actions of mapping classes on deformation spaces. The modular group $\mathrm{Mod}(S)$ acts on $\mathrm{Teich}(S)$ and $AH(S)$ by precomposing the marking by φ^{-1}. The action on $AH(S)$ restricts to $QF(S) = \mathrm{Teich}(S) \times \mathrm{Teich}(\overline{S})$ by acting simultaneously on each factor. Our primary concern will be with the action of the modular group on Bers' slice B_Y obtained by letting φ act on the first factor.

Structure theory of hyperbolic 3-manifolds. By a theorem of P. Scott, any 3-manifold M with finitely generated fundamental group contains a *compact core* $\mathcal{M} \subset M$: a compact submanifold $\mathcal{M}$ of M whose inclusion is a homotopy equivalence [Scott]. A theorem of McCullough [McC, Thm. 2] gives a relative version.

Let $M_{(r,R)}$ denote the subset of M where the injectivity radius $\mathrm{inj}\colon M \to \mathbb{R}^+$ lies in (r, R). For ϵ less than a universal ϵ_3 each component T of the *Margulis ϵ-thin part* $M_{(0,\epsilon)}$ has a standard form: T is either a *Margulis tube*, a solid torus neighborhood of a short geodesic, or a *cusp*, the quotient of a horoball B in $\mathbb{H}^3$ by a $\mathbb{Z}$, or $\mathbb{Z} \oplus \mathbb{Z}$ parabolic action stabilizing B. The *cuspidal thin part* of $P(M)$ of M is the components of $M_{(0,\epsilon_3)}$ corresponding to cusps of M (see e.g. [BP, Thm. D.3.13]).

Definition 2.11 *Let M be a complete hyperbolic 3-manifold with finitely generated fundamental group and cuspidal thin part P. A* relative compact core $(\mathcal{M}, \mathcal{P})$ *for $M - P$ relative to P is a smoothly embedded compact submanifold $\mathcal{M} \subset M - P$ with a collection $\mathcal{P} \subset \partial\mathcal{M}$ of compact incompressible annuli and tori called its* parabolic locus *such that each component of $\partial\mathcal{M} - \mathrm{int}(\mathcal{P})$ has negative Euler characteristic, and*

1. *$\mathcal{M} \cap \partial P = \mathcal{P}$,*
2. *the inclusion $\iota\colon (\mathcal{M}, \mathcal{P}) \to (M - P, \partial P)$ is a homotopy equivalence,*
3. *for each component P_s of P there is a component $\mathcal{P}_{t(s)}$ of $\mathcal{P}$ such that $\iota(\mathcal{P}_{t(s)})$ lies in ∂P_s.*

We denote by $\partial_0\mathcal{M}$ the complement $\partial\mathcal{M} - \mathrm{int}(\mathcal{P})$ of the interior of the parabolic locus in $\partial\mathcal{M}$. Call $M - P$ the pared submanifold *of M.*

We state the resulting decomposition of M.

Decomposition 2.12 *The pared submanifold $M - P$ decomposes into a relative compact core $(\mathcal{M}, \mathcal{P})$ relative to P, and a finite collection of* ends $\{E_m\}_{m=1}^p$ *of $(M - P)$ meeting $\mathcal{M}$ in surfaces $S_m \subset \partial_0 \mathcal{M}$.*

Remark: As has become customary (cf. [Th1] [Mc2]), we refer to as "ends" of $M - P$ the *neighborhoods* E_m of the ends of the topological space $M - P$.

When the end E_m has finite volume intersection with the convex core of M, it is called *geometrically finite* and *geometrically infinite* otherwise. The following is well known (see [EM] or [Min, Thm. 5.2]).

Theorem 2.13 *The quasi-isometry type of M depends only on the topology of its relative compact core $\iota\colon (\mathcal{M}, \mathcal{P}) \to (M - P, \partial P)$, its parabolic locus $\mathcal{P}$, and the quasi-isometry types of its geometrically infinite ends E_m marked by $\iota|_{S_m}$.* $\square$

We call the intersection $(M - P) \cap \mathrm{core}(M)$ the *pared submanifold of the convex core of M*. The following is an evident consequence.

Corollary 2.14 *The quasi-isometry type of M depends only on the quasi-isometry type the pared submanifold of the convex core of M.* $\square$

The following theorem recasts of a theorem of Marden [Mar, Prop. 5.4]:

Theorem 2.15 *Let $(f\colon S \to M) \in \partial B_Y$. Let the curves $\{\gamma_j\}_{j=1}^k \subset S$ represent its accidental parabolics. Then decomposition 2.12 determines a relative compact core $\iota\colon (S \times I, \mathcal{P}) \to M - P$, where*

$$\mathcal{P} = \left(\cup_{j=1}^k A_j \times \{0\} \right) \cup (\partial S \times I)$$

is a collection of annuli such that A_j has core curve γ_j. Incompressible surfaces $S_Y = S \times \{1\}$ and $S_m \times \{0\} \subset S \times \{0\}$, $m = 1, \ldots, p$, make up $\partial(S \times I) - \mathrm{int}(\mathcal{P})$ and cut off ends E_Y and $\{E_m\}_{m=1}^p$ of the pared submanifold with the following properties.

1. **The fixed end.** *The surface S_Y cuts off a geometrically finite end E_Y asymptotic to $Y \subset \partial M$, $\iota_*(\pi_1(S_Y)) = f_*(\pi_1(S))$ stabilizes a unique invariant component Ω_Y, with $\Omega_Y / f_*(\pi_1(S)) \cong Y$.*
2. **Geometrically finite ends.** *The surface S_m cuts off a geometrically finite end E_m if and only if $\iota_*(\pi_1(S_m)) = \Gamma_m$ is quasi-Fuchsian. For each component $\Omega \subset \Omega(\Gamma_m)$, $X_m = \Omega / \Gamma_m \cong \mathrm{int}(S_m)$ and E_m is asymptotic to $X_m \subset \partial M$.*
3. **Geometrically infinite ends.** *S_m cuts off a geometrically infinite end E_m if and only if $\iota_*(\pi_1(S_m)) = \Gamma_m$ is a totally degenerate group. Furthermore, $\Omega(\Gamma_m)/\Gamma_m \cong \mathrm{int}(S_m)$.* $\square$

When Γ_m is quasi-Fuchsian, we call the end E_m *quasi-Fuchsian*.

3 Pleated surfaces

Definition 3.1 *A* pleated surface *is a triple* (g, X, N) *consisting of a surface* $X \in$ Teich(S)*, a hyperbolic 3-manifold* N *and a* pleated mapping $g\colon X \to N$*:*

1. g *sends rectifiable arcs in* X *to rectifiable arcs in* N *of the same length,*
2. g *is incompressible, and*
3. *for each* $x \in X$*,* g *maps some geodesic segment through* x *isometrically.*

The *pleating locus* $\mathcal{L}(g)$ of the map g, the set of points $x \in X$ where g fails to be a local isometry, is a geodesic lamination on X (see [Th2, Prop. 5.1]).

Definition 3.2 *Let* $f\colon S \to N$ *be an incompressible map of* S *into* N*. Then* $\mathcal{PS}(f)$*, the* marked pleated surfaces homotopic to f*, consists of pairs* (g, X) *of surfaces* $(h\colon S \to X) \in$ Teich(S) *and pleated mappings* $g\colon X \to N$*, so that* $g \circ h \simeq f$*.*

We give $\mathcal{PS}(f)$ the following topology: a sequence $\{(g_n, X_n)\} \to (g, X)$ if there are marking-preserving quasi-isometries $q_n\colon X \to X_n$ with quasi-isometry constants $L(q_n) \to 1$, such that $g_n \circ q_n$ converges uniformly to g (cf. [Th1, 8.8.1] [CEG, 5.2.14]). We say the incompressible map f is *internally non-parabolic* if $f_*(\gamma)$ is hyperbolic whenever $\gamma \in \pi_1(S)$ is non-peripheral. We will make use of the following theorem of Thurston (see [CEG, Thm. 5.2.18]).

Theorem 3.3 (Thurston) *Let* $K \subset N$ *be a compact subset of* N*, and* $f\colon S \to N$ *a continuous incompressible map that is internally non-parabolic. Let* $\{(g_n, X_n)\}$ *be a sequence of pleated surfaces in* $\mathcal{PS}(f)$ *whose images* $g_n(X_n)$ *all meet* K*. Then* $\{(g_n, X_n)\}$ *has a convergent subsequence, or* N *has a finite cover* $\widetilde{N}$ *that fibers over the circle.*

Given a pleated surface (g, X) in $\mathcal{PS}(f)$, the pleating locus $\mathcal{L}(g)$ determines an element in $\mathcal{GL}(S)$ via the implicit marking on $X \in$ Teich(S). The map

$$\mathcal{L}\colon \mathcal{PS}(f) \to \mathcal{GL}(S)$$

that assigns to each pleated surface (g, X) its pleating locus $\mathcal{L}(g) \in \mathcal{GL}(S)$ is continuous (see [Th1, Prop. 8.10.4] [CEG, Lemma 5.3.2]).

Notice that g maps leaves of $\mathcal{L}(g)$ isometrically. More generally:

Definition 3.4 *Let* $f\colon S \to N$ *be incompressible. A geodesic lamination* $\lambda \in \mathcal{GL}(S)$*, is* realizable *in* N *in the homotopy class of* f *if there is a pleated surface* $(g, X) \in \mathcal{PS}(f)$ *so that* g *restricts to each leaf of* λ *on* X *by a local isometry.*

Via its implicit marking $f\colon S \to M$, any $M \in AH(S)$ determines a subset $\mathcal{R}(M) \subset \mathcal{ML}(S)$ of measured laminations whose support is realizable in M the homotopy class of f. We say $\mu \in \mathcal{R}(M)$ is *realizable* in M as well. The set $\mathcal{R}(M)$ is dense in $\mathcal{ML}(S)$, and if f is internally non-parabolic definition then $\mathcal{R}(M)$ is open [CEG, 5.3.10].

Thurston proved [Bon1, Prop. 4.5] that there is a unique continuous function

$$\text{length}\colon \text{Teich}(S) \times \mathcal{ML}(S) \to \mathbb{R}$$

whose restriction to $\text{Teich}(S) \times (\mathbb{R}^+ \times \mathcal{S})$ satisfies $\text{length}_X(t\gamma) = t\ell_X(\gamma^*)$ where $\ell_X(.)$ denotes arclength on X and γ^* is the geodesic representative of $\gamma \in \mathcal{S}$ on X.

When $\mu \in \mathcal{R}(M)$ is realizable by a pleated surface (g, X) in $\mathcal{PS}(f)$, we define $\text{length}_M(\mu)$ to be $\text{length}_X(\mu)$. Given $(M, \mu) \in AH(S) \times \mathcal{ML}(S)$ let $\mathrm{R}_M(\mu)$ be the maximal sublamination μ' realizable in M. Let $\underline{\text{length}}\colon AH(S) \times \mathcal{ML}(S) \to \mathbb{R}$ denote the function

$$(M, \mu) \to \text{length}_M(\mathrm{R}_M(\mu)).$$

Then length is the restriction of $\underline{\text{length}}$ to the set $\mathfrak{R} \subset AH(S) \times \mathcal{ML}(S)$ of pairs (M, μ) for which μ is realizable $\overline{\text{in } M.}$ In [Br2, Thm. 7.1] we prove:

Theorem 3.5 LENGTH CONTINUOUS. *The function* $\underline{\text{length}}$ *is continuous.*

We will use following corollary in our applications:

Corollary 3.6 [Br2, Cor. 7.3] *Let pairs* $\{(M_n, \mu_n)\} \subset \mathfrak{R}$, *converge to the pair* (M, μ) *in* $AH(S) \times \mathcal{ML}(S)$. *If* $\text{length}_{M_n}(\mu_n) \to 0$ *then* μ *is non-realizable in* M.

Tameness. Give the pared submanifold of a hyperbolic manifold M the standard decomposition into its relative compact core $\mathcal{M}$ and ends E_m (decomposition 2.12).

Definition 3.7 *An end* E *of the pared submanifold is* simply degenerate *if there is a sequence* $\{c_n\}$ *of non-peripheral simple closed curves in the boundary component* $\iota(S)$ *of the relative compact core cutting off* E *whose geodesic representatives* c_n^* *exit every compact subset of the end* E.

Simply degenerate ends are topologically products [Th1, Ch. 9] [Bon1]. The hyperbolic manifold M is *geometrically tame* if all ends E_m of its pared submanifold are *geometrically tame*: i.e. either geometrically finite or simply degenerate.

Theorem 3.8 (Thurston, Bonahon) *Each* $M \in AH(S)$ *is geometrically tame;* M *is homeomorphic to* $\text{int}(S) \times \mathbb{R}$.

4 Iteration on a Bers slice

Fix $(X, Y) \in \text{Teich}(S) \times \text{Teich}(\overline{S})$, and a mapping class φ. Let

$$Q_i = Q(\varphi^i X, Y) \in AH(S)$$

denote the sequence obtained from iteration of φ on the Bers slice B_Y, and let Q_φ denote any accumulation point of the sequence $\{Q_i\}_{i=1}^\infty$, with its implicit marking $f\colon S \to Q_\varphi$. By [Mc2, Thm 3.7] [Bers4, Lem. 1], φ is *quasi-isometrically realized* on Q_φ: the quasi-isometric distance $d(\varphi(Q_\varphi), Q_\varphi)$ is bounded.

We say a sequence in a Bers slice *converges up to marking-preserving quasi-isometry* to Q if for any accumulation point Q_∞ we have $d(Q, Q_\infty) < \infty$. Since the Teichmüller distance $d_T(\varphi^{i+k}X, \varphi^i X) = d_T(\varphi^k X, X)$ is independent of i, there are (marking-preserving) uniformly quasi-conformal conjugacies between the uniformizing Kleinian groups for Q_i and Q_{i+k} for each i. These conjugacies extend to equivariant uniform quasi-isometries between $\widetilde{Q_i}$ and $\widetilde{Q_{i+k}}$ (see e.g. [Mc2, Thm. 2.5]) so the quasi-isometric distance

$$d(Q(\varphi^i X, Y), Q(\varphi^{i+k} X, Y))$$

is uniformly bounded independent of i. Thus, lower semi-continuity of $d(.,.)$ [Mc2, Prop. 3.1] implies that for a convergent subsequence $\{Q_{i_j}\}_{j=1}^{\infty} \to Q_\varphi$ and any k, the sequence $\{Q_{i_j+k}\}_{j=1}^{\infty}$ converges up to marking-preserving quasi-isometry to Q_φ.

A choice of baseframe ω_φ in the limit manifold Q_φ marked by f determines a Kleinian group $f_*(\pi_1(S))$, with

$$Q_\varphi = \mathbb{H}^3 / f_*(\pi_1(S)).$$

In this section we determine the full decomposition of $f_*(\pi_1(S))$ in the sense of theorem 2.15. We show that this decomposition depends only on φ.

Combinatorics. Let $(\Pi(\varphi), S_{\mathsf{F}}(\varphi), S_{\mathsf{P}}(\varphi))$ be the Nielsen-Thurston decomposition of S for φ (see theorem 2.6).

Decomposition 4.1 *Let $\varphi \in \mathrm{Mod}(S)$, and let Q_φ be an accumulation point in $\overline{B_Y}$ of the iteration $\{Q(\varphi^i X, Y)\}_{i=1}^{\infty} \subset B_Y$. Then*

1. *the minimal reducing partition $\Pi(\varphi)$ is the set of accidental parabolics for Q_φ,*
2. *each component of $S_{\mathsf{P}}(\varphi)$ corresponds to a degenerate cover of Q_φ, and*
3. *each component of $S_{\mathsf{F}}(\varphi)$ corresponds to a quasi-Fuchsian cover of Q_φ.*

This is a succinct summary of a series of assertions (proposition 4.2 through theorem 4.7); a more detailed version is given in decomposition 4.8.

For reference, we choose pairwise disjoint representatives of the curves and subsurfaces of the Nielsen-Thurston decomposition $(\Pi(\varphi), S_{\mathsf{F}}(\varphi), S_{\mathsf{P}}(\varphi))$ which we will denote by the same names.

By the above remarks, the set of all marking-preserving quasi-isometry classes of accumulation points of $\{Q_i\}$ is identified with the set of marking-preserving quasi-isometry classes of accumulation points of $\{Q_{ki}\}$. Since the above description of the parabolics, quasi-Fuchsian covers, and degenerate covers of Q_φ is preserved under marking-preserving quasi-isometries, it suffices to give the above description for the accumulation points of $\{Q_{ki}\}$ for any k. To simplify the discussion, therefore, we pass to a stable iterate of φ that also stabilizes the isotopy classes of each connected component of the subsurfaces $S_{\mathsf{F}}(\varphi)$ and $S_{\mathsf{P}}(\varphi)$.

Proposition 4.2 *Let $\gamma \in \mathcal{S}$ be an accidental parabolic for Q_φ. Then*

I. *$i(\gamma, \delta) = 0$ for each $\delta \in \Pi(\varphi)$, and*
II. *$i(\gamma, \eta) = 0$ for each essential isotopy class η of simple closed curves in $S_{\mathsf{P}}(\varphi)$.*

Proof. By theorem 2.15, the isotopy classes of accidental parabolics for Q_φ themselves form a partition Π_P of S (see [Msk1, Lem. 2] or [Mar, Prop. 5.4]). Since

φ is quasi-isometrically realized on Q_φ (see [Mc2, Thm 3.7] [Bers4, Lem. 1]), φ preserves isotopy classes of parabolics in Q_φ. Thus $\varphi(\Pi_P) = \Pi_P$.

By irreducibility, a pseudo-Anosov mapping class preserves no non-peripheral isotopy class of simple closed curves. Therefore γ is not a non-peripheral isotopy class of a curve in any component $S_0 \subset S_{\mathsf{P}}(\varphi)$. Likewise, by a surgery argument on S, if $i(\gamma, \partial S_{\mathsf{P}}(\varphi))$ is non-zero, φ cannot preserve γ. This proves (II) since if $i(\gamma, \eta) \neq 0$ either γ is isotopic into $S_{\mathsf{P}}(\varphi)$ or $i(\gamma, \partial S_{\mathsf{P}}(\varphi)) \neq 0$.

Lastly, if $\delta \in \Pi(\varphi)$ is an isolated Dehn twisting curve for φ, then φ cannot preserve γ if $i(\gamma, \delta) \neq 0$. By corollary 2.9, every isotopy class in $\Pi(\varphi)$ is represented either by a Dehn-twisting curve for φ or a component of $\partial S_{\mathsf{P}}(\varphi)$. It follows that $i(\gamma, \eta)$ is zero for each $\eta \in \Pi(\varphi)$, proving (I). $\square$

We now verify part (2) of decomposition 4.1.

Theorem 4.3 *If S_0 is a component of $S_{\mathsf{P}}(\varphi)$, then $f_*(\pi_1(S_0))$ is totally degenerate.*

Proof. Again, it suffices to prove the theorem for an iterate of φ that stabilizes the isotopy class of S_0 by a pseudo-Anosov mapping class $\psi_0 \in \mathrm{Mod}(S_0)$.

Lemma 4.4 *Let $\varphi \in \mathrm{Mod}(S)$ have the property that $\varphi(\mu) = c\mu$, $c > 1$, $\mu \in \mathcal{ML}(S)$. Then we have $\lim_{i\to\infty} \mathrm{length}_{Q_i}(\mu) = 0$.*

Proof. By a theorem of Bers, for each $\gamma \in \mathcal{S}$ we have

$$\mathrm{length}_{Q(X,Y)}(\gamma) \leq 2\min\{\mathrm{length}_X(\gamma), \mathrm{length}_Y(\gamma)\}$$

(see [Bers2, Thm. 3] [Mc1, Prop. 6.4]). Continuity of $\mathrm{length}_X(.)$ on $\mathcal{ML}(S)$ implies the same inequality holds for lengths of measured laminations. Since we have $\mathrm{length}_{\varphi X}(\nu) = \mathrm{length}_X(\varphi^{-1}(\nu))$ for all $\nu \in \mathcal{ML}(S)$ and $X \in \mathrm{Teich}(S)$, we have

$$\mathrm{length}_{Q_i}(\mu) \leq 2\mathrm{length}_{\varphi^i(X)}(\mu) = (2/c^i)\mathrm{length}_X(\mu).$$

Thus, $\mathrm{length}_{Q_i}(\mu)$ tends to zero as $i \to \infty$. $\square$

Let ψ_0 have unstable lamination $\mu^u \in \mathcal{ML}(S_0)$. Since $\psi_0(\mu^u) = c\mu^u$ for some $c > 1$, it follows from the above lemma and corollary 3.6 (or [Bon3, Thm. D]) that μ^u is non-realizable in Q_φ.

By corollary 2.9 and proposition 4.2, a connected component T of the complementary subsurface determined up to isotopy by $S - \Pi_P$ contains S_0 up to isotopy. The subgroup $f_*(\pi_1(T))$ cannot be quasi-Fuchsian since μ^u lies in $\mathcal{ML}(T)$ and is non-realizable [CEG, Thm. 5.3.11] [Th1, Prop. 8.7.7]. Since the only elements $\beta \in \pi_1(T)$ of $\pi_1(T)$ with parabolic image $f_*(\beta)$ are peripheral in T, it follows that $f_*(\pi_1(T))$ is totally degenerate by theorem 2.15.

We claim that T is isotopic to S_0. It suffices to show that each boundary component of S_0 is parabolic in Q_φ. Assume on the contrary that $\gamma \subset \partial S_0$ has geodesic representative γ^* in Q_φ. Let weighted simple closed curves $\{t_n\zeta_n\} \subset \mathcal{ML}(S_0)$ converging to μ^u determine geodesics ζ_n^* in Q_φ.

Pleated surfaces $\{(g_n, X_n)\} \subset \mathcal{PS}(f|_T)$ realizing $\gamma \cup t_n\zeta_n$ have images $g_n(X_n)$ that all intersect the compact set γ^*. The map $f|_T$ is internally non-parabolic, so by theorem 3.3, continuous variance of the pleating locus, and openness of the realizable laminations in $\mathcal{ML}(T)$ ([CEG, 5.3.2., 5.3.10]), we may pass to a convergent subsequence converging to a limit pleated surface (g_∞, X_∞) that realizes the limit

$$\lim_{n\to\infty} \gamma \cup t_n\zeta_n = \gamma \cup \mu^u$$

in $\mathcal{ML}(T)$, contradicting non-realizability of μ^u. It follows that γ is parabolic in Q_φ, and so ∂S_0 consists of parabolics. Hence, S_0 is isotopic to T, and $f_*(\pi_1(S_0))$ is totally degenerate. □

The next two theorems verify part (1) of decomposition 4.1.

Theorem 4.5 *Each isotopy class in $\Pi(\varphi)$ is an accidental parabolic.*

Proof. Let $\delta \in \Pi(\varphi) - \partial S_{\mathsf{P}}(\varphi)$. By corollary 2.9, φ Dehn-twists about δ. Since we have shown each isotopy class in $\partial S_{\mathsf{P}}(\varphi)$ is an accidental parabolic for Q_φ, it suffices to prove that δ is an accidental parabolic.

Let $\eta \in \mathcal{S}$ be such that $i(\eta, \delta) \neq 0$ and $i(\eta, \Pi(\varphi) - \delta) = 0$. Consider the sequence

$$t_i\varphi^i(\eta) \quad \text{where} \quad t_i = 1/\text{length}_X(\varphi^i(\eta))$$

of length-1 measured laminations on X. As $\varphi^i(\eta)$ winds more and more around δ, any $\zeta \in \mathcal{S}$ has intersection number $i(\zeta, t_i\varphi^i(\eta))$ tending to $i(\zeta, \delta)/\text{length}_X(\delta)$. Thus, the sequence $\{t_i\varphi^i(\eta)\}$ tends to the lamination $\delta/\text{length}_X(\delta)$ in $\mathcal{ML}(S)$ as i tends to infinity.

Furthermore, we have

$$\text{length}_{Q_i}(t_i\varphi^i(\eta)) \leq 2\text{length}_{\varphi^i X}(t_i\varphi^i(\eta)) = 2t_i\text{length}_X(\eta)$$

which tends to 0 as $i \to \infty$. Thus, by corollary 3.6, δ is non-realizable in Q_φ, and we conclude that δ is parabolic in Q_φ. (One may also argue as in [KT, §3]). □

Theorem 4.6 *The partition Π_P by accidental parabolics of Q_φ and the minimal reducing partition $\Pi(\varphi)$ for φ are identical.*

Proof. By theorem 4.5, we have $\Pi(\varphi) \subset \Pi_P$. Let F_0 be a component of $S_{\mathsf{F}}(\varphi)$. By theorem 4.3 it suffices to prove that any accidental parabolic in F_0 is peripheral.

Let $\gamma \in \mathcal{S}$ be an accidental parabolic in F_0. By stability, we have $\varphi(\gamma) = \gamma$ and $\varphi(\delta) = \delta$). If γ is not peripheral in F_0, there is a $\delta \in \mathcal{S}$ lying in F_0 so that $i(\delta, \gamma) \neq 0$. Since γ is parabolic, and the accidental parabolics form a partition of S, δ has positive length L in Q_φ. By continuity of the length of δ on $AH(S)$, $\text{length}_{Q_i}(\delta)$ remains bounded below throughout the sequence.

Since $\varphi(\delta) = \delta$, $\text{length}_{\varphi^i X}(\delta)$ is constant as i tends to infinity. By a theorem of Sullivan, generalized by Epstein and Marden [Sul1, Prop. 1] [EM, Lem. 2.3.1] there is a universal K so that we have

$$\text{length}_{\partial_h Q_i}(\delta) \leq K\text{length}_{\partial_c Q_i}(\delta)$$

and likewise for $\overline{\partial}_h Q_i$ and $\overline{\partial}_c Q_i$.

Let $(\delta_h)_i$ and $(\overline{\delta}_h)_i$ denote the unique geodesic representatives in the free homotopy class of δ on $\partial_h(Q_i)$ and $\overline{\partial}_h(Q_i)$. Let $(\delta^*)_i$ be the geodesic representative of δ in Q_i. Since $\text{length}_{\varphi^i X}(\delta)$ is constant, both lengths

$$\text{length}_{\partial_h(Q_i)}(\delta) \quad \text{and} \quad \text{length}_{\overline{\partial}_h(Q_i)}(\delta)$$

remain uniformly bounded above throughout the sequence. Construct pleated cylinders C_i in $\text{core}(Q_i)$ representing the free homotopy class of δ so that

1. $\partial C_i = (\delta_h)_i \cup (\overline{\delta}_h)_i$, and
2. $(\delta^*)_i \subset C_i$

(cf. the *pleated annulus* construction of [Th4, §3]).

Since the lengths of $(\delta_h)_i$ and $(\overline{\delta}_h)_i$ remain bounded, and the length of $(\delta^*)_i$ remains bounded *below*, the diameter of each cylinder C_i remains uniformly bounded by some $K' > 0$ throughout the sequence. Since $i(\gamma, \delta) \neq 0$, it follows from [Bon1, Lem. 3.3] that the unique geodesic $(\gamma^*)_i$ in the free homotopy class of γ in Q_i must intersect C_i for each i. Since $\text{length}_{Q_i}(\gamma)$ tends to 0 as i tends to ∞, $(\gamma^*)_i$ is arbitrarily deep in the Margulis thin part $(Q_i)_{(0,\epsilon_3)}$ where ϵ_3 is the 3-dimensional Margulis constant (see e.g. [BP, Thm. D.3.13]). Once the depth is greater than K', both C_i and $(\gamma^*)_i$ lie in the same component of $(Q_i)_{(0,\epsilon_3)}$, which violates the thick-thin decomposition since γ and δ do not commute in $\pi_1(S)$. Thus, γ is peripheral in F_0, and $\Pi_P = \Pi(\varphi)$. $\square$

Finally, we verify part (3) of decomposition 4.1.

Theorem 4.7 *If F_0 is a component of $S_{\mathsf{F}}(\varphi)$, then $f_*(\pi_1(F_0))$ is quasi-Fuchsian.*

Proof. By the previous theorem, an isotopy class γ of essential simple closed curves in F_0 is parabolic if and only if it is peripheral in F_0. Hence, the restriction $f_*|_{\pi_1(F_0)}$ determines an element of $AH(F_0)$ with no accidental parabolics. By theorem 2.15, $f_*(\pi_1(F_0))$ is either totally degenerate or quasi-Fuchsian.

Let δ be a non-peripheral isotopy class in F_0; then δ is non-parabolic in Q_φ, and thus $\text{length}_{Q_\varphi}(\delta) = L > 0$. Construct uniformly bounded diameter pleated cylinders C_i through $(\delta^*)_i$ as in the proof of theorem 4.6.

If $f_*(\pi_1(F_0))$ were totally degenerate, tameness (theorem 3.8) implies there would be a sequence $c_j \subset F_0$ of essential simple closed curves, non-peripheral in F_0 whose geodesic representatives leave every compact subset of $\text{core}(Q_\varphi)$. Let $Q(F_0) = \mathbb{H}^3/f_*(\pi_1(F_0))$, and let pleated surfaces $(g_j, X_j) \in \mathcal{PS}(f|_{F_0})$ realize c_j in $Q(F_0)$. Applying [Bus, Thm. 5.2.6] there is a uniform constant B so that given $X \in \text{Teich}(F_0)$ we can always find a *maximal* partition of F_0 all of whose elements have length less than B on X. Thus there are simple closed curves $d_j \in \mathcal{S}(F_0)$ so that $\text{length}_{X_j}(d_j) < B$ for each j, and either $d_j = c_j$ or the geodesic representatives of d_j and c_j on X_j intersect. The curves d_j assume an infinite number of distinct isotopy classes since otherwise some curve d_{j_0} would have bounded length representatives in $Q(F_0)$ with arbitrarily large separation, forcing d_{j_0} to be parabolic.

There are constants $t_j \to 0$ so that $t_j d_j$ tends to a non-zero lamination $\nu \in \mathcal{ML}(S)$ after passing to a further subsequence. But $\text{length}_{Q(F_0)}(t_j d_j) < t_j B \to 0$, so ν is non-realizable in $Q(F_0)$, by corollary 3.6. We claim that $i(\delta, d_j) > 0$ for j sufficiently large: otherwise, since $f|_{F_0}$ is internally non-parabolic, a sequence of pleated surfaces realizing δ and d_j has a convergent subsequence in $\mathcal{PS}(f|_{F_0})$ (by theorem 3.3, as in the proof theorem 4.3) whose limit realizes ν.

Thus geodesic representatives $(d_j)^*$ of d_j in any Q_i intersect the cylinder C_i for j sufficiently large. Since $\text{length}_{Q_i}(d_j)$ tends to $\text{length}_{Q_\varphi}(d_j)$ for each j, given $a \in \mathbb{Z}^+$ there is an N_a so that for all $i > N_a$, we have

$$\text{length}_{Q_i}(d_j) < 2B$$

for all $j \leq a$. But the diameter of C_i is uniformly bounded, and the number of homotopically distinct simple closed curves of length bounded by $2B$ that intersect a set of bounded diameter in a hyperbolic 3-manifold is bounded. Therefore, $f_*(\pi_1(F_0))$ is not totally degenerate, so it is quasi-Fuchsian. □

Accumulation points. We assemble in one place the structural information about Q_φ. Let S_m, $m = 1, \ldots, p$ denote connected components of $S_{\mathsf{P}}(\varphi)$ and let F_n, $n = 1, \ldots, q$ denote connected components of $S_{\mathsf{F}}(\varphi)$.

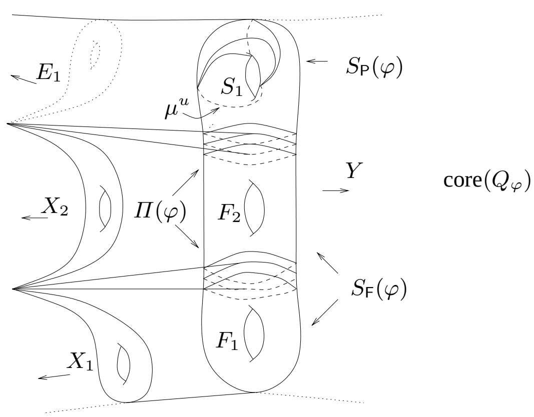


Figure 5. The convex core of a limit Q_φ of iteration of φ on B_Y.

Decomposition 4.8 *Let φ be a stable mapping class. Let $(f\colon S \to Q_\varphi)$ be any accumulation point of $\{Q(\varphi^i X, Y)\} \subset B_Y$ with cuspidal thin part P_φ. Then φ determines the following decomposition of the pared submanifold $Q_\varphi - P_\varphi$.*

1. *The* **relative compact core** $\iota\colon (\mathcal{M}, \mathcal{P}) \to (Q_\varphi - P_\varphi, \partial P_\varphi)$ *has parabolic locus* $\mathcal{P} = ((\partial S) \times I) \cup \left(\overline{\mathcal{N}(\Pi(\varphi))} \times \{0\}\right)$ *and relative boundary*

$$\partial_0 \mathcal{M} = (S_{\mathsf{P}}(\varphi) \times \{0\}) \cup (S_{\mathsf{F}}(\varphi) \times \{0\}) \cup (S \times \{1\}).$$

2. *There is a geometrically finite* **fixed end** E_Y *of* $Q_\varphi - P_\varphi$ *asymptotic to* $Y \subseteq \partial\overline{Q_\varphi}$ *cut off by* $\iota(S \times \{1\})$ *such that* $\iota_*(\pi_1(S \times \{1\})) = f_*(\pi_1(S))$.
3. *Each component of* $\iota(S_{\mathsf{F}}(\varphi) \times \{0\})$ *cuts off a* **quasi-Fuchsian end** *of* $Q_\varphi - P_\varphi$ *asymptotic to a surface* $X_n \subset \partial\overline{Q_\varphi}$.
4. *Each component of* $\iota(S_{\mathsf{P}}(\varphi) \times \{0\})$ *cuts off a* **simply degenerate end** *of* $Q_\varphi - P_\varphi$.

5 Asymptotic geometry

In the case of iteration of a pseudo-Anosov mapping class ψ on a Bers' slice, simply degenerate ends arising in the limit are *asymptotically isometric* to a standard model depending only on ψ (see [Mc2, §3.5]). For a general stable mapping class φ, we show that a similar analysis holds where φ induces a pseudo-Anosov mapping class ψ on a subsurface. This gives a model for the quasi-isometry type of an accumulation point of $\{Q(\varphi^i X, Y)\}_{i=1}^\infty$ that depends only on φ.

Three-manifolds fibering over the circle. Given $\psi \in \mathrm{Mod}(S)$, its *mapping torus* is given by the identification

$$T_\psi = S \times I/(x,0) \sim (\psi(x), 1).$$

The manifold T_ψ fibers over the circle S^1 with monodromy map ψ by projection of each surface (S,t) to t. Its orientation is given as the product of the orientation on S with the orientation of I. Thurston proved the following remarkable theorem [Th4].

Theorem 5.1 (Thurston) *Given* $\psi \in \mathrm{Mod}(S)$*, the mapping torus* T_ψ *is hyperbolic if and only if* ψ *is pseudo-Anosov.*

The pseudo-Anosov mapping class ψ canonically determines an element $M_\psi \in AH(S)$ by passing to the cover corresponding to $\iota_*(\pi_1(S))$ where $\iota \colon S \to T_\psi$ is the inclusion of a fiber.

In [Mc2], McMullen gives a construction of the hyperbolic structure on T_ψ based on the iteration $Q(\psi^i X, Y)$ of ψ on a Bers' slice B_Y. A consequence is the convergence of such iteration ([Mc2, Thm. 3.11]), which follows from a classification of the new degenerate ends in any pair of algebraic accumulation points up to quasi-isometry. In this section we relate this discussion to the *induced* pseudo-Anosov mapping classes of a general mapping class. Our main tool will be the following theorem [Mc2, Thm. 3.17] which characterizes points in $AH(S)$ that tend to M_ψ under iteration of ψ on $AH(S)$ in terms of the asymptotic geometry of their degenerate ends.

Theorem 5.2 (McMullen) *Let* Q *belong to* $AH(S)$*. Then* $\psi^n(Q) \to M_\psi$ *if and only if the negative end of the pared submanifold of* Q *admits an asymptotic isometry to that of* M_ψ *compatible with markings.*

In our setting, the theorem asserts the existence of a marking and orientation preserving diffeomorphism $h \colon E \to E_\psi$ from the negative end of the pared submanifold of Q to that of M_ψ so that for any k and any $\epsilon > 0$, there is a compact set

$K \subset E$ such that h is ϵ-close to an isometry in the C^k topology on $E - K$. (cf. [Mc2, pp. 55]). In particular, h is a quasi-isometry on E.

To distinguish the negative and positive ends of the pared submanifold of $M \in AH(S)$, we remark that by tameness (see theorem 3.8) M is homeomorphic to $\text{int}(S) \times \mathbb{R}$. By requiring that this homeomorphism preserve orientation, we may label the ends of the pared submanifold of M "positive" and "negative" corresponding to the positive and negative ends of $\mathbb{R}$. In particular, when S is closed and $Q(X, Y)$ thus has no cusps, the negative end of $Q(X, Y)$ is asymptotic to X and the positive end of $Q(X, Y)$ is asymptotic to Y.

Asymptotic geometry. For iteration of a stable mapping class φ, decomposition 4.8 gives a correspondence between components $S_0 \subset S_{\mathsf{P}}(\varphi)$ of the pseudo-Anosov subsurface for φ, and simply degenerate ends E of the accumulation point $(f \colon S \to Q_\varphi) \in B_Y$ of iteration of φ on B_Y. Passing to the least iterate of φ that leaves each component of $S_{\mathsf{P}}(\varphi)$ invariant, consider the action of the pseudo-Anosov mapping class $\psi \in \text{Mod}(S_0)$ induced by φ on S_0. Let $\mu^s,\ \mu^u \in \mathcal{ML}(S_0)$ be the stable and unstable laminations for ψ. By decomposition 4.8 the subgroup $f_*(\pi_1(S_0))$ of $f_*(\pi_1(S))$ is totally degenerate, and μ^u is non-realizable in Q_φ.

Let E_0 be the degenerate end of $Q_\varphi - P_\varphi$ cut off by the surface $\iota(S_0 \times \{0\})$ as in decomposition 4.8. Let $T_{\psi^{-1}}$ denote the mapping torus for ψ^{-1} and $M_{\psi^{-1}} \in AH(S_0)$ its cover corresponding to the fiber. Let $P_{\psi^{-1}}$ denote the cuspidal thin part $M_{\psi^{-1}}$. Then the negative end of $M_{\psi^{-1}} - P_{\psi^{-1}}$ gives a model for E_0.

Theorem 5.3 *The end E_0 of $Q_\varphi - P_\varphi$ is asymptotically isometric to the negative end of $M_{\psi^{-1}} - P_{\psi^{-1}}$.*

Proof. Consider the cover $M = \mathbb{H}^3 / f_*(\pi_1(S_0))$ corresponding to $f_*(\pi_1(S_0))$. The marking on Q_φ naturally determines a marking on M, which in turn determines an element of $AH(S_0)$ since $f_*(\beta)$ is parabolic for each boundary component β in ∂S_0. Denote by $M_0 = (f_0 \colon S_0 \to M)$ this cover marked by a homotopy equivalence f_0 such that $(f_0)_* = f_*|_{\pi_1(S_0)}$, with its cuspidal thin part P_0.

The degenerate end E_0 of $Q_\varphi - P_\varphi$ lifts isometrically to the negative end of the pared submanifold $M_0 - P_0$ of M_0 (see figure 6). By theorem 5.2, then, it suffices

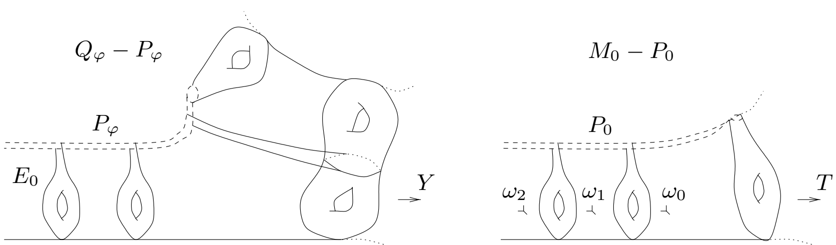


Figure 6. The cover corresponding to the end E_0.

to show that M_0 converges to $M_{\psi^{-1}}$ under iteration of ψ^{-1} on $AH(S_0)$.

Theorem 5.4 *The sequence $\{\psi^{-n}(M_0)\}$ converges to $M_{\psi^{-1}}$ in $AH(S_0)$.*

Before proving theorem 5.4, we discuss a central tool in its proof.

Compactness. We must first show that the sequence $\psi^{-n}(M_0) = M_n$ ranges in a compact subset of $AH(S_0)$. The case when $\varphi = \psi^{-1}$ is discussed in [Mc2, §3]: by Thurston's double limit theorem [Th4, Thm. 4.1] (see also [Otal1, Thm. 5.0.1], [Can1, Thm. 6.1]) the quasi-Fuchsian manifolds $Q(\psi^i X, \psi^{-j} Y)$, $i, j > 0$, range in a precompact subset of $AH(S)$. Since each M_n is a limit of $Q(\psi^{i-n} X, \psi^{-n} Y)$ as i tends to ∞, M_n lies in the compact closure $\overline{Q(\psi^i X, \psi^{-j} Y)} \subset AH(S)$, $i, j > 0$. Thus the sequence $\{M_n\}$ converges up to subsequence.

Our situation differs slightly in that while each M_n covers a limit of quasi-Fuchsian groups, it is not itself given as such a limit. We resort to the following internal formulation of the double limit theorem ([Th4, Thm. 6.3]): a pair of measured laminations $\mu, \nu \in \mathcal{ML}(S)$ *binds* S if for every $\gamma \in \mathcal{S}$ we have

$$i(\mu, \gamma) + i(\nu, \gamma) > 0.$$

Theorem 5.5 (Thurston) *If $\{M_n\}$ is a sequence in $AH(S)$ of marked hyperbolic 3-manifolds and $\mu_n \to \mu$ and $\nu_n \to \nu$ are sequences of measured laminations such that*

$$\text{length}_{M_n}(\mu_n) \quad \textit{and} \quad \text{length}_{M_n}(\nu_n)$$

remain bounded, then if μ and ν bind the surface S, there is a subsequence of $\{M_n\}$ that converges.

Corollary 5.6 COMPACTNESS. *The sequence $M_n = \psi^{-n}(M_0)$ ranges in a compact subset of $AH(S_0)$.*

Proof (of corollary 5.6). We exhibit sequences $\{\mu_n\}$ and $\{\nu_n\}$ in $\mathcal{ML}(S_0)$ tending to μ^u and μ^s whose lengths in M_n remain bounded. This suffices, since μ^u and μ^s bind the surface S_0.

Let $\partial M_0 = T \in \text{Teich}(S_0)$. Then we have $\partial M_n = \psi^{-n} T$. As in lemma 4.4

$$\text{length}_{\psi^{-n}T}(\mu^s) = \frac{1}{c^n}\text{length}_T(\mu^s),$$

for some $c > 1$. Since T is incompressible, continuity of length together with [EM, Lem. 2.3.1] (cf. [Sul1, Prop. 1]) guarantees that there is a universal constant K such that

$$K\text{length}_{\psi^{-n}T}(\mu^s) \geq \text{length}_{M_n}(\mu^s).$$

It follows that $\text{length}_{M_n}(\mu^s)$ tends to zero as n tends to infinity.

Note that M_0, and hence each M_n, has no accidental parabolics. Let $\{t_i \zeta_i\}$ be a sequence of weighted simple closed curves tending to μ^u. Since μ^u is non-realizable in M_0, ζ_i^* exits every compact subset of $\text{core}(M_0)$. Otherwise, a subsequence of pleated surfaces realizing ζ_i would converge to a pleated surface realizing μ^u by theorem 3.3, as in the proof of theorem 4.3. Then by continuity of length, theorem 3.5, $\{\text{length}_{M_0}(t_i \zeta_i)\}$ tends to zero as i tends to infinity (see also [Otal1, Thm. 6.2.11], [Th1, Prop. 9.3.4]). Since μ^u is preserved by ψ up to scale, it is non-realizable in every M_n. Thus, for any fixed constant $C > 0$ and any n there is an i_n so that $t_{i_n}\zeta_{i_n}$ has $\text{length}_{M_n}(t_{i_n}\zeta_{i_n}) < C$. The sequences $\mu_n = t_{i_n}\zeta_{i_n}$ and $\nu_n \equiv \mu^s$ have bounded length in M_n, and converge to laminations μ^u and μ^s that bind S. Thus, by theorem 5.5, $\{M_n\}$ converges after passing to a subsequence. $\square$

Convergence to the fiber. We now prove theorem 5.4. The argument mirrors the construction of the hyperbolic structure on the mapping torus T_ψ, which appears in [Mc2, §3.4], [Otal1, Ch. 6], and in [Th4].

Proof (of theorem 5.4). Pass to a subsequence of $\{M_n\}$ converging to a limit $M_\infty \in AH(S_0)$. Choose convergent lifts (M_n, ω_n) to $AH_\omega(S_0)$ converging to $(M_\infty, \omega_\infty)$. Such lifts determine representations $\rho_n \colon \pi_1(S_0) \to \mathrm{PSL}_2(\mathbb{C})$ converging on generators to a limit ρ_∞. Note that M_n are different markings of the same manifold M_0, and the ω_n are frames in M_0.

We claim the baseframes ω_n travel arbitrarily deeply into the convex core of M_0. Let η be a non-peripheral element of $\pi_1(S_0)$. By algebraic convergence, the translation distance of $\rho_n(\eta)$ at ω_n is uniformly bounded by $L > 0$. This means there is a path $(\eta)_n \subset M_n$ based at ω_n representing η of length bounded by L. But the paths $(\eta)_n$ represent *distinct* homotopy classes $\psi^n(\eta)$ in the manifold M_0. Hence, ω_n leaves every compact subset $K \subset M_0$ as there is a uniform bound $J_{K,L}$ to the number of closed curves in distinct homotopy classes with length at most L in K. By algebraic convergence, ω_n lies in $(M_0)_{(r,R)}$ for some $r > 0$ and $R < \infty$, so it follows that the baseframes ω_n travel arbitrarily deeply into the convex core. Thus, the limit sets $\Lambda(\rho_n(\pi_1(S)))$ converge to $\widehat{\mathbb{C}}$ in the Hausdorff topology (see [Mc2, Prop. 2.3]).

Since M_0 lies in $AH(S)$, it is geometrically tame, and we have $\text{inj}(x) < R$ for each $x \in \text{core}(M_0)$ (see [Can2, Thm. 6.2] [Bon1]). We pass to a geometrically convergent subsequence of $\{(M_n, \omega_n)\}$ converging geometrically to a limit (N, ω). Let $\mathbb{H}^3$ uniformize the geometric limit (N, ω) as the quotient $(\mathbb{H}^3, \widetilde{\omega})/\Gamma_G$ by the Kleinian group Γ_G. By compactness in the geometric topology and continuous variance of the limit sets [Mc2, Prop. 2.4] the injectivity radius of N is bounded by R throughout $\text{core}(N)$, and the limit set $\Lambda(\Gamma_G)$ is all of $\widehat{\mathbb{C}}$.

The quasi-isometric realization of φ^{-1} on Q_φ lifts to a quasi-isometry $\Theta \colon M_0 \to M_0$ in the homotopy class of ψ^{-1}. For each n, there is a lift

$$\widetilde{\Theta_n} \colon \widetilde{M_n} \to \widetilde{M_n}$$

whose uniformly quasi-conformal extension (by [Mc2, Thm. 2.5]) $\overline{\Theta_n} \colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ induces the precomposition by ψ_*^{-1} of ρ_n: in other words, for each n we have

$$\overline{\Theta_n} \cdot \rho_n(g) \cdot \overline{\Theta_n}^{-1} = \rho_n(\psi_*^{-1}(g)).$$

Choose three non-peripheral non-commuting elements $\eta_\kappa \in \pi_1(S_0)$, $\kappa = 1, 2, 3$. The attracting fixed points of $\rho_n(\eta_\kappa)$ in $\widehat{\mathbb{C}}$ are distinct for each n, and converge to the distinct attracting (or possibly parabolic) fixed points of $\rho_\infty(\eta_\kappa)$, $\kappa = 1, 2, 3$, since ρ_∞ is discrete and faithful. The uniformly quasi-conformal conjugacies $\overline{\Theta_n}$ map the triple of attracting fixed points of $\rho_n(\eta_\kappa)$ to the triple of attracting fixed points of $\rho_n(\psi_*^{-1}(\eta_\kappa))$, so it follows that they have a quasi-conformal limit $\overline{\Theta_\infty}$ after passing to a subsequence [Gard, §1.8, Lem. 6].

The limit $\overline{\Theta_\infty}$ has the property that

$$\overline{\Theta_\infty} \cdot \rho_\infty(g) \cdot \overline{\Theta_\infty}^{-1} = \rho_\infty(\psi_*^{-1}(g)).$$

Moreover, any element $\gamma \in \Gamma_G$ is the limit of a sequence $\{\rho_n(g_n)\}_{n=1}^\infty$ for some sequence $\{g_n\}_{n=1}^\infty \subset \pi_1(S_0)$. Since we have

$$\overline{\Theta_n} \cdot \rho_n(g_n) \cdot \overline{\Theta_n}^{-1} = \rho_n(\psi_*^{-1}(g_n))$$

it follows that $\rho_n(\psi_*^{-1}(g_n))$ also converges in $\mathrm{PSL}_2(\mathbb{C})$ with limit $\gamma' \in \Gamma_G$. Thus

$$\overline{\Theta_\infty} \cdot \Gamma_G \cdot \overline{\Theta_\infty}^{-1} = \Gamma_G.$$

It follows that there is a quasi-isometry $\beta\colon N \to N$ covered by a quasi-isometry $\widetilde{\beta}\colon M_\infty \to M_\infty$ in the homotopy class of ψ^{-1}: i.e. so that $\widetilde{\beta} \circ f_\infty \simeq f_\infty \circ \psi^{-1}$, by extending $\overline{\Theta_\infty}$ to $\mathbb{H}^3$.

Since $\Lambda(\Gamma_G) = \widehat{\mathbb{C}}$, and $\mathrm{inj}(x) < R$ for each $x \in \mathrm{core}(N)$, the group Γ_G is quasi-conformally rigid (see [Mc2, Thm. 2.9], [Sul2]). Hence $\overline{\Theta_\infty}$ is conformal. The induced isometry $\xi\colon N \to N$ lifts to an isometry $\alpha\colon M_\infty \to M_\infty$ in the homotopy class of ψ^{-1} so the quotient $M_\infty/\langle\alpha\rangle$ is a hyperbolic manifold homotopy equivalent to $T_{\psi^{-1}}$. By a theorem of Stallings [St], $M_\infty/\langle\alpha\rangle \cong T_{\psi^{-1}}$ and $M_\infty = M_{\psi^{-1}}$. □

Continuation of the proof of theorem 5.3. By theorem 5.4, the sequence $\{\psi^{-n}(M_0)\}$ converges to $M_{\psi^{-1}}$ in $AH(S_0)$, so we are finished. □

Algebraic limits. Having analyzed simply degenerate ends arising from iteration, we are ready to prove the following theorem.

Theorem 5.7 QUASI-ISOMETRIC ALGEBRAIC LIMITS. *Let φ be an element of $\mathrm{Mod}(S)$, and let Q_φ and Q'_φ be any pair of accumulation points of the iteration $\{Q(\varphi^i X, Y)\}_{i=1}^\infty \subset B_Y$. Then there is a quasi-isometry $\Theta\colon Q_\varphi \to Q'_\varphi$ compatible with markings.*

Proof. When φ has finite order s, the quasi-Fuchsian manifolds $\{Q(\varphi^i X, Y)\}_{i=1}^s$ are mutually quasi-isometric in a manner compatible with markings.

Let φ have infinite order. Let P_φ and P'_φ be the cuspidal thin parts of Q_φ and Q'_φ respectively. By decomposition 4.8, $Q_\varphi - P_\varphi$ and $Q'_\varphi - P'_\varphi$ admit relative compact cores with a common model $(\mathcal{M}, \mathcal{P})$ up to diffeomorphism given by $\mathcal{M} \cong S \times I$ and $\mathcal{P} = \overline{\mathcal{N}(\Pi(\varphi))} \times \{0\} \cup ((\partial S) \times I)$. Choose inclusions

$$\iota\colon (\mathcal{M}, \mathcal{P}) \to (Q_\varphi - P_\varphi, \partial P_\varphi) \quad \text{and} \quad \iota'\colon (\mathcal{M}, \mathcal{P}) \to (Q'_\varphi - P'_\varphi, \partial P'_\varphi).$$

When $S_{\mathsf{P}}(\varphi) \neq \varnothing$, there is a correspondence between simply degenerate ends E_m and E'_m, $m = 1, \ldots, p$, of $Q_\varphi - P_\varphi$ and $Q'_\varphi - P'_\varphi$ each cut off by $\iota(S_m \times \{0\})$ and $\iota'(S_m \times \{0\})$ for each component $S_m \subset S_{\mathsf{P}}(\varphi)$.

After passing to an iterate of φ that stabilizes each component S_m, let $\psi_m \in \mathrm{Mod}(S_m)$ be the pseudo-Anosov mapping class induced on S_m. By theorem 5.4, E_m and E'_m are each asymptotically isometric to the negative end of the pared submanifold of $M_{\psi_m^{-1}}$ in a manner compatible with marking.

Since Q_φ and Q'_φ have relative compact cores with the same topology and parabolic locus, and their corresponding simply degenerate ends admit marking-preserving quasi-isometries, by theorem 2.13 there is a marking-preserving quasi-isometry

$$\Theta \colon Q_\varphi \to Q'_\varphi.$$

□

6 Geometric limits and convergence

In this section we show that iteration of a stable mapping class on a Bers slice converges algebraically and geometrically.

By analogy with the finite order case, we go on to validate the hypothesis of stability by showing that whenever the complex dimension of Teich(S) is greater than one (cf. corollary 6.11) there exists $\varphi \in \mathrm{Mod}(S)$ and $X \in$ Teich(S) so that the iteration $\{Q(\varphi^i X, Y)\}_{i=1}^\infty$ has more than one algebraic accumulation point.

Convergence theorems. Our main goal in this section is to prove:

Theorem 6.1 STABLE ITERATION CONVERGES. *Let $\varphi \in \mathrm{Mod}(S)$ be a stable mapping class. Then the sequence $\{Q(\varphi^i X, Y)\}_{i=1}^\infty$ converges algebraically to a limit Q_φ and geometrically to a limit N covered by Q_φ.*

The theorem is established for pseudo-Anosov mapping classes in [Mc2, Thm. 3.11], [CT, §7]. The algebraic and geometric limits agree, and they do not depend on X. We will first establish the following lemma.

Lemma 6.2 *Let $\varphi \in \mathrm{Mod}(S)$ be a mapping class. Let Q_φ and Q'_φ be any pair of accumulation points of the sequence $\{Q(\varphi^i X, Y)\}_{i=1}^\infty$ in $\overline{B_Y}$. Let N and N' be geometric accumulation points covered by Q_φ and Q'_φ after passing to further subsequences.*

Then there is a quasi-isometry $\Theta \colon N \to N'$ that is covered by a quasi-isometry $\widetilde{\Theta} \colon Q_\varphi \to Q'_\varphi$ compatible with markings.

As in the finite order case, we have the following corollary of theorem 6.1.

Corollary 6.3 *For any mapping class $\varphi \in \mathrm{Mod}(S)$, the number of algebraic and geometric accumulation points of the sequence $\{Q(\varphi^i X, Y)\}$ is bounded by the stable power s for φ.*

(See definition 2.8; recall that the stable power is bounded in terms of S).

Outline of the proof. We outline our approach to theorem 6.1. By a re-marking trick, we may build any geometric limit N directly by a gluing (combination) of algebraic limits Q_φ of $Q_i = Q(\varphi^i X, Y)$ and $Q_{\varphi^{-1}}$ of $\varphi^{-i}(Q_i) = Q(X, \varphi^{-i} Y)$ along their common quasi-Fuchsian ends. This is justified by a general *gluing lemma* (lemma 6.5) which describes a process by which covers of the geometric limit N of a sequence $M_i \in AH(S)$ may be built from algebraic limits of different markings $f_i \colon S \to M_i$ and $g_i \colon S \to M_i$ of the manifolds M_i. After gluing Q_φ and $Q_{\varphi^{-1}}$ we have a manifold that covers N, and by consideration of the domains of discontinuity (proposition 6.6 and corollary 6.7) we find the covering is a homeomorphism.[4]

Quasi-isometries between pairs N and N' of geometric accumulation points are constructed by gluing together marking-preserving quasi-isometries between pairs of algebraic accumulation points of $\{Q_i\}$ and $\{\varphi^{-i}(Q_i)\}$, proving lemma 6.2. Consideration of the conformal boundary shows $\partial N = X \sqcup Y = \partial N'$, so by rigidity these quasi-isometries are homotopic to isometries. This proves algebraic and geometric convergence (theorem 6.1).

We briefly introduce new terminology to discuss gluing hyperbolic manifolds, prove the gluing lemma, and go on to prove lemma 6.2 from which theorem 6.1 follows. To discuss the gluing process, we allow the complete oriented hyperbolic 3-manifold M to be disconnected.

The quasi-isometric deformation space. A complete oriented hyperbolic 3-manifold M has a *quasi-isometric deformation space* $\text{Def}(M)$ consisting of pairs (h, N) of hyperbolic 3-manifolds N marked by quasi-isometries

$$h \colon M \to N$$

up to isometries preserving orientation and marking. The quasi-isometric distance $d(.,.)$, defined analogously to that on $AH(S)$, determines the topology on $\text{Def}(M)$. When ∂M is incompressible, work of Ahlfors, Bers, Mostow, Prasad, Kra, Maskit and Sullivan culminates in a fundamental parametrization for $\text{Def}(M)$; in particular by a theorem of Sullivan, provided either $\pi_1(M)$ is finitely generated ([Sul2, Thm. V.]), or the injectivity radius is bounded on $\text{core}(M)$ ([Mc2, Thm. 2.9]), we have

$$\text{Def}(M) = \text{Teich}(\partial M)$$

via the natural projection (e.g. [Kra, Thm. 14]; see also [Th2, Thm. 1.3]).

The skinning map. Let M be a complete oriented hyperbolic 3-manifold, with conformal boundary ∂M. Assume M is not itself quasi-Fuchsian, and let $\partial_{\text{qf}} M \subset \partial M$ denote the subset consisting of components $W \subset \partial M$ whose corresponding covers are quasi-Fuchsian. The *skinning map*

$$\sigma \colon \text{Teich}(\partial_{\text{qf}} M) \to \text{Teich}(\overline{\partial_{\text{qf}} M})$$

is defined by passing to each quasi-Fuchsian cover and recording the structure of the new conformal boundary component. While the natural covering $Q(W, \sigma(W)) \to M$, extends to an embedding on W, the surface $\sigma(W)$ does not embed (unless $M = Q(W, \sigma(W))$), and its structure depends on the structures on all surfaces in ∂M simultaneously.

[4] At this point, the homeomorphism type of N (theorem 1) can be deduced, but we defer this to §7.

The gluing problem. Let $G \subset \partial_{\rm qf} M$ be a portion of the quasi-Fuchsian conformal boundary for M so that for each component M_0 of M, we have $G \cap \partial M_0 \neq \varnothing$. An orientation reversing fixed-point-free involution $\tau\colon G \to G$ determines an isometry $\tau\colon \text{Teich}(\overline{G}) \to \text{Teich}(G)$. Then τ determines a *gluing problem* for the deformation space $\text{Def}(M)$ of M, which seeks a fixed point for the composition

$$\tau \circ \sigma|_G\colon \text{Teich}(G) \to \text{Teich}(G)$$

(cf. [Mc1, §3.3] [Mor, §9] [Otal2]). Once found, a fixed point x for $\tau \circ \sigma|_G$, solves the gluing problem as follows: if

$$p_\tau\colon \text{Def}(M) \to \text{Teich}(G)$$

is the natural projection, then any manifold $M' \in \{p_\tau^{-1}(x)\}$ has quasi-Fuchsian ends corresponding to G that are isometrically compatible with the gluing data τ. Then the *gluing* M'/τ of M' by τ is the complete (connected) hyperbolic 3-manifold obtained by isometrically joining together the quasi-Fuchsian ends that correspond under the involution τ; M' is called a *solution* to the gluing problem for the gluing data τ.

The existence and standard properties of M'/τ follow from the *Klein-Maskit combination theorems:* for example, [AC, Thms. 8.1, 8.2] (we refer the reader to [Msk2, Thms. VII.C.1, VII.E.5] for more detailed versions). While M'/τ is not literally a gluing of the Kleinian manifold $\overline{M'}$ along its conformal boundary, the complete hyperbolic manifold M'/τ is realized as an isometric gluing along embedded surfaces truncating the quasi-Fuchsian ends corresponding to G.

Compatible covering spaces. Let M be a solution to a gluing problem $\tau\colon G \to G$, for $G \subset \partial_{\rm qf} M$. For each component $W \subset G$, the quasi-Fuchsian manifold $Q(W, \sigma(W))$ determines a covering space of M. That M is a solution implies that the two covering spaces $Q(W, \sigma(W))$ and $Q(\tau \circ \sigma(W), \sigma(W))$ are isometrically identified as the quasi-Fuchsian manifold Q, which covers M by two distinct coverings

$$q_1\colon Q \to M \quad \text{and} \quad q_2\colon Q \to M.$$

The covering map q_1 extends to an embedding $\overline{q_1}\colon W \hookrightarrow \partial M$ and the covering map q_2 extends to an embedding $\overline{q_2}\colon \sigma(W) \hookrightarrow \partial M$.

When τ identifies more than one pair of surfaces, we allow Q to be a disconnected union of quasi-Fuchsian covering spaces of M (one component for each pair of surfaces identified by τ) and let $q_1\colon Q \to M$ and $q_2\colon Q \to M$ be covering maps defined as above on each component of Q.

A choice of baseframe $\omega \in Q$, naturally identifies a (quasi-Fuchsian) Kleinian group $\Gamma(Q)$ uniformizing the component of Q containing ω, and the images $q_1(\omega)$ and $q_2(\omega)$ identify Kleinian groups $\Gamma_1(M)$ and $\Gamma_2(M)$ uniformizing the connected component of M containing $q_\kappa(\omega)$ so that the standard frame at the origin in $\mathbb{H}^3$ lies over $q_\kappa(\omega)$, $\kappa = 1, 2$.

The groups $\Gamma(Q) < \Gamma_1(M), \Gamma_2(M)$ satisfy the hypotheses of the first combination theorem ([AC, Thm. 8.1]) if $q_1(\omega)$ and $q_2(\omega)$ lie in different components of

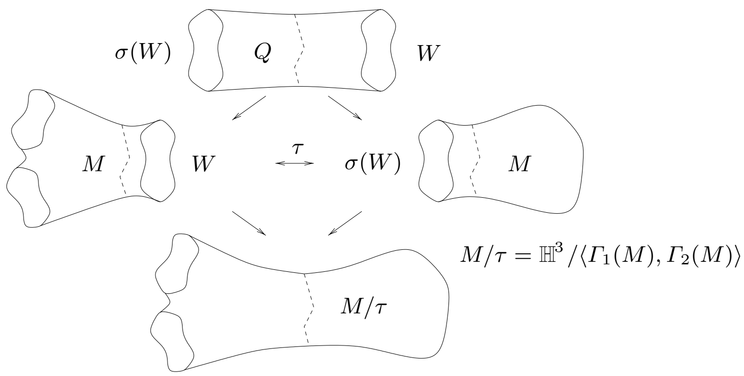


Figure 7. Isometrically gluing hyperbolic manifolds.

M and the second combination theorem ([AC, Thm. 8.2]) if they lie in the same component. The gluing M/τ is then $\mathbb{H}^3/\Gamma_c$, where Γ_c is the result of successive *combinations* and *HNN-extensions* of $\Gamma_1(M)$ and $\Gamma_2(M)$ along the quasi-Fuchsian subgroups $\Gamma(Q)$ determined by choosing a baseframe ω in each component of Q (see also [Br3, App. B], [Mc1, §3.3]).

Using the combination theorems, the following facts about gluings are easily verified. Let M solve the gluing problem for τ. Then we have:

GI If $G \subset \partial_{\mathrm{qf}} M$ is the domain for τ, then $\partial(M/\tau) = \partial M - G$.

GII Each component of M covers M/τ by a local isometry.

GIII If N is another hyperbolic 3-manifold, and there are locally isometric covering maps

$$Q \xrightarrow[q_2]{q_1} M \xrightarrow[p_2]{p_1} N$$

that are *compatible* with τ: i.e. $p_1 \circ q_1 = p_2 \circ q_2$ and $\tau \circ \overline{q_1}|_W$ is isotopic to $\overline{q_2} \circ \sigma|_W$, then the gluing M/τ covers N by a local isometry.

The following theorem concerning gluings seems to be well-known:

Theorem 6.4 *Let M be a solution to the gluing problem determined by τ, and let $M' \in \mathrm{Def}(M)$ be any other solution. Then there is a quasi-isometry*

$$\Theta\colon M/\tau \to M'/\tau$$

so that any lift $\widetilde{\Theta}\colon M \to M'$ is marking-preserving.

Proof (sketch). We sketch a proof of theorem 6.4 (see also [Br3, App. B] for more detail). Let F be the (in general disconnected) surface with one component for each pair of surfaces to be identified by the gluing map τ (with the previous notation $F = G/\tau$). By work of Maskit [Msk2] and Anderson-Canary [AC, Lem. 3.1, 6.3] there is a properly embedded *gluing surface* $s\colon F \to M/\tau$ so that s is incompressible on each component F_n of F and so that, $(s|_{F_n})_*(\pi_1(F_n)) = (p_1 \circ q_1|_{Q_n})_*(\pi_1(Q_n))$

where $Q_n \subset Q$ is the quasi-Fuchsian component of Q corresponding to F_n (the image $s(F)$ is the quotient of a *system of spanning disks*, one for each quasi-Fuchsian subgroup involved in the combination). Moreover, s may be taken to be totally geodesic on neighborhoods of the ends of F.

The map s lifts to a map $\tilde{s}\colon F \to Q$ whose image separates Q into two disjoint sets E_1 and E_2 so that q_1 is an embedding on E_1 and q_2 is an embedding on E_2 (cf. figure 7). Likewise, p_1 is an embedding on $q_1(E_2)$, p_2 is an embedding on $q_2(E_1)$, and M/τ is the isometric identification

$$M/\tau = (M - q_1(E_1)) \cup_{\hat{\tau}} (M - q_2(E_2))$$

where $\hat{\tau}\colon q_1(\tilde{s}(F)) \to q_2(\tilde{s}(F))$ is defined by $\hat{\tau}(q_1(\tilde{s}(x))) = q_2(\tilde{s}(x))$.

Let $s'\colon F \to M'/\tau$ be the gluing surface for M'/τ with lift $\tilde{s}'\colon F \to Q'$ and let $q_1'\colon Q' \to M'$ and $q_2'\colon Q' \to M'$ be the corresponding covering maps. Let Θ_0 be the natural marking-preserving quasi-isometry $\Theta_0\colon M \to M'$; perturb Θ_0 so it restricts to a quasi-isometry of pared submanifolds of M and M'.

There is an ϵ so that the cuspidal parts P_ϵ and P'_ϵ of $M_{(0,\epsilon)}$ and $M'_{(0,\epsilon)}$ intersect $s(F)$ and $s'(F)$ in totally geodesic cylinders asymptotic to cusps. Taking $M - P_\epsilon$ and $M' - P'_\epsilon$ as our pared submanifolds insures that the intersections of $q_1(\tilde{s}(F)) \sqcup q_2(\tilde{s}(F))$ and $q_1'(\tilde{s}'(F)) \sqcup q_2'(\tilde{s}'(F))$ with the pared submanifolds of M and M' are surfaces cutting off corresponding quasi-Fuchsian ends. After precomposition of s' by an isotopy, we may adjust Θ_0 by a homotopy in a compact neighborhood of $s(F)$ to obtain a quasi-isometry Θ_0' so that $\Theta_0'(q_1(\tilde{s}(x))) = q_1'(\tilde{s}'(x))$ and $\Theta_0'(q_2(\tilde{s}(x))) = q_2'(\tilde{s}'(x))$.

It follows that Θ_0' respects the gluings M/τ and M'/τ: Θ_0' determines a quasi-isometry $\Theta\colon M/\tau \to M'/\tau$ of the gluings, and by construction any lift $\widetilde{\Theta}\colon M \to M'$ of Θ to M is marking preserving. $\square$

The gluing lemma. The following lemma should be a useful tool for studying algebraic and geometric limits of general sequences in $AH(S)$.

Lemma 6.5 THE GLUING LEMMA. *Let $T \subset S$ be a connected, essential, proper subsurface of negative Euler characteristic. Let*

$$(f_i\colon S \to M_i) \to (f\colon S \to M) \quad \textit{and} \quad (g_i\colon S \to M_i) \to (g\colon S \to M')$$

be convergent sequences in $AH(S)$ such that $(f_i)_*|_{\pi_1(T)}$ is conjugate to $(g_i)_*|_{\pi_1(T)}$ in $\pi_1(M_i)$ for each i. Then any geometric limit N of M_i covered by $p\colon M \to N$ is also covered by $p'\colon M' \to N$, so that $p_* \circ f_*|_{\pi_1(T)} = p'_* \circ g_*|_{\pi_1(T)}$ up to conjugacy.

If, moreover,

a) the limit $f_*(\pi_1(T))$ is quasi-Fuchsian, and
b) $Q(W,Z) = \mathbb{H}^3/f_*(\pi_1(T))$ covers M and M' by covering maps that extend to embed W in ∂M and Z in $\partial M'$,

then $M \sqcup M'$ solves the natural gluing problem with data $\tau\colon W \rightleftharpoons Z$ such that $\tau_* = id_{\pi_1(T)}$, and the covers $Q(W,Z) \to M \to N$ and $Q(W,Z) \to M' \to N$ are compatible with τ.

Proof. We first verify that we may pass to a subsequence to extract a geometric limit N that is covered by both M and M'. Recall from the proof of proposition 2.3 that the geometric limit covered by M is obtained via choosing baseframes $\omega_i \in M_i$ that determine convergent lifts to $AH_\omega(S)$: the based manifolds (M_i, ω_i) determine Kleinian groups Γ_i and the markings $f_i \colon S \to (M_i, \omega_i)$ determine an algebraically convergent sequence of representations $\rho_i \colon \pi_1(S) \to \mathrm{PSL}_2(\mathbb{C})$ (with image Γ_i) that converge to a limit $\rho_\infty = f_*$ on each $\alpha \in \pi_1(S)$.

Since $(f_i)_*|_{\pi_1(T)}$ is conjugate to $(g_i)_*|_{\pi_1(T)}$ and $\rho_i(\pi_1(T))$ is non-elementary, the markings $g_i \colon S \to (M_i, \omega_i)$ also determine algebraically convergent representations $\varrho_i \colon \pi_1(S) \to \mathrm{PSL}_2(\mathbb{C})$ (also with image Γ_i) tending to $\varrho_\infty = g_*$. As in proposition 2.3, we may pass to a geometrically convergent subsequence of (M_i, ω_i) converging to a hyperbolic manifold (N, ω). The geometric limit N is naturally covered by M and M'. Let

$$p \colon M \to N \quad \text{and} \quad p' \colon M' \to N$$

denote the locally isometric covering projections.

We verify that the images $p_* \circ f_*(\pi_1(T))$ and $p'_* \circ g_*(\pi_1(T))$ are conjugate in $\pi_1(N)$. Let Γ_G be the Kleinian uniformization of (N, ω).

Choose any three non-peripheral non-commuting elements η_κ, $\kappa = 1, 2, 3$, in $\pi_1(T)$. Then there are elements $b_i \in \pi_1(M_i)$ such that

$$(f_i)_*(\eta_\kappa) = b_i \circ (g_i)_*(\eta_\kappa) \circ b_i^{-1}$$

for each i and each κ. The elements $\rho_i(\eta_\kappa)$ of $\mathrm{PSL}_2(\mathbb{C})$ are conjugate by $\beta_i \in \Gamma_i$ to $\varrho_i(\eta_\kappa)$, for $\kappa = 1, 2, 3$. Since ρ_i and ϱ_i converge on each η_κ, the attracting (or parabolic) fixed points of $\rho_i(\eta_\kappa)$ are mapped by each β_i to the attracting (or parabolic) fixed points for $\varrho_i(\eta_\kappa)$. The fixed points of $\rho_i(\eta_\kappa)$ converge in $\widehat{\mathbb{C}}$ to the distinct fixed points of $\rho_\infty(\eta_\kappa)$ and likewise for the fixed points of $\varrho_i(\eta_\kappa)$, so the elements β_i converge to an element β that conjugates the fixed points of $\rho_\infty(\eta_\kappa)$ to the fixed points of $\varrho_\infty(\eta_\kappa)$. Since $\beta_i \in \Gamma_i$, it follows that β is contained in the geometric limit Γ_G. Hence, $p_* \circ f_*|_{\pi_1(T)}$ and $p'_* \circ g_*|_{\pi_1(T)}$ are conjugate in $\pi_1(N)$.

Assuming *a)* and *b)* hold in addition, we may pass to the cover corresponding to $p_* \circ f_*(\pi_1(T))$ to obtain the quasi-Fuchsian manifold $Q(W, Z) \in QF(T)$. By hypothesis *b)*, the locally isometric covering map $Q(W, Z) \to N$, factors through coverings to M and M' that embed W in ∂M and Z in $\partial M'$. Evidently, $M \sqcup M'$ solves the gluing problem determined by the natural gluing involution

$$\tau \colon W \rightleftharpoons Z$$

satisfying $\tau_* = id_{\pi_1(T)}$. Moreover, the locally isometric coverings

$$Q(W, Z) \to M \to N \quad \text{and} \quad Q(W, Z) \to M' \to N$$

are compatible with τ, and thus, the gluing $(M \sqcup M')/\tau$ covers N by a local isometry. $\square$

Carathéodory convergence. We wish to apply the gluing lemma (lemma 6.5) to the setting of iteration to prove lemma 6.2. To control the ways the algebraic limit can cover the geometric limit, we recall certain facts about their conformal boundaries.

Combining continuous variance of the limit sets in the geometric topology [KT, Cor. 2.2] with the *Carathéodory convergence theorem* [Du, Thm. 3.1], Kerckhoff and Thurston analyze algebraic and geometric convergence in a Bers slice in terms of the sphere at infinity (see [KT, Prop. 2.3]): in the Carathéodory topology on *open* subsets of $\widehat{\mathbb{C}}$, a sequence of open subsets $\Omega_i \subset \widehat{\mathbb{C}}$ converges to an open subset $\Omega \subset \widehat{\mathbb{C}}$ if and only if the complements $(\widehat{\mathbb{C}} - \Omega_i)$ converge to $(\widehat{\mathbb{C}} - \Omega)$ in the Hausdorff topology (see [JM, 4.1] [KT, Prop. 2.3]).

Proposition 6.6 CARATHÉODORY CONVERGENCE. *Let $Q_i \in \overline{B_Y}$ converge algebraically to Q_∞ and geometrically to N covered by Q_∞. Let convergent lifts $(Q_i, \omega_i) \to (Q_\infty, \omega_\infty)$ to $AH_\omega(S)$ with geometric limit (N, ω) determine Kleinian groups Γ_i, Γ_∞ and Γ_G respectively. Let $\Omega_+(\Gamma_i) \subset \Omega(\Gamma_i)$ cover $Y \subset \partial Q_i$ and let $\Omega_-(\Gamma_i) = \Omega(\Gamma_i) - \Omega_+(\Gamma_i)$. Then*

1. *the limit sets $\Lambda(\Gamma_i)$ converge to $\Lambda(\Gamma_G)$ in the Hausdorff topology,*
2. *after passing to a subsequence so that $\Omega_+(\Gamma_{i_j}) \to \Omega_+$ and $\Omega_-(\Gamma_{i_j}) \to \Omega_-$, we have $\Omega_+ \cap \Omega_- = \varnothing$, $\Omega(\Gamma_G) = \Omega_+ \sqcup \Omega_-$, and Ω_+ and Ω_- are Γ_G-invariant,*
3. *the component $\Omega_Y \subset \Omega(\Gamma_\infty)$ covering $Y \subset \partial \overline{Q_\infty}$ embeds in $\Omega_+ \subset \Omega(\Gamma_G)$.*

Proof. Assertion (1) is proven in [KT, Cor. 2.2] for quasi-Fuchsian groups and follows from [Mc2, Prop. 2.4] and [Can2, Thm. 6.2] in general.

For assertion (2), if x lies in $\Omega_+ \cap \Omega_-$, then x lies in $\Omega_+(\Gamma_{i_j}) \cap \Omega_-(\Gamma_{i_j})$ for all $i_j \gg 0$, which contradicts the fact that $\Omega_+(\Gamma_i) \cap \Omega_+(\Gamma_i) = \varnothing$ for all i. Thus $\Omega_+ \cap \Omega_- = \varnothing$. If x lies in $\Omega(\Gamma_G)$, then x has an open neighborhood that lies in infinitely many $\Omega_+(\Gamma_{i_j})$ or infinitely many $\Omega_-(\Gamma_{i_j})$ so x lies in $\Omega_+ \sqcup \Omega_-$. Likewise, if x lies in Ω_+ then x lies in $\Omega_+(\Gamma_{i_j}) \subset \Omega(\Gamma_{i_j})$ for all $i_j \gg 0$, so x lies in $\Omega(\Gamma_G)$, and similarly for $x \in \Omega_-$. Thus $\Omega_G = \Omega_+ \sqcup \Omega_-$. If y lies in Ω_+ then it has a neighborhood contained in $\Omega_+(\Gamma_{i_j})$ for all $i_j \gg 0$. The same holds for $x \in \Omega_-$, provided Ω_- is non-empty. If there were an element $\gamma \in \Gamma_G$ for which $\gamma(y) = x$ it would be the limit of elements $\gamma_{i_j} \in \Gamma_{i_j}$ with the property that $\gamma_{i_j}(y_{i_j}) = x$ with $y_{i_j} \to y$ contradicting the Γ_{i_j}-invariance of $\Omega_+(\Gamma_{i_j})$ and $\Omega_-(\Gamma_{i_j})$. Thus Ω_+ and Ω_- are each Γ_G-invariant and assertion (2) follows.

For assertion (3), since the Q_i range in $\overline{B_Y}$, Ω_Y is the image $w_\infty(\Delta)$ of the unit disk Δ under the locally uniform limit w_∞ of a sequence of univalent maps $w_i \colon \Delta \to \widehat{\mathbb{C}}$ with $w_i(\Delta) = \Omega_+(\Gamma_i)$ (see [Bers2], [KT, §2]). By the Carathéodory convergence theorem [Du, Thm. 3.1], $w_\infty(\Delta)$ is a component of Ω_+. □

This proposition has the following consequence in the quotients (cf. [JM]).

Corollary 6.7 *Let $Q_i \to Q_\infty$ be a convergent sequence in $\overline{B_Y}$. Then for any geometric limit N covered by Q_∞, the covering map $\pi \colon Q_\infty \to N$ extends to a holomorphic embedding on $Y \subset \partial \overline{Q_\infty}$.*

Proof. By proposition 6.6, the covering map π extends to the finite area Riemann surface Y by a locally isometric covering map $\overline{\pi} \colon Y \to Z$ for some $Z \subset \partial N$. Thus, $\overline{\pi}$ is finite to one.

We claim $\overline{\pi}$ is an embedding. Choose baseframes determining Kleinian groups as in proposition 6.6. If $\overline{\pi}$ is not an embedding, then the algebraic limit Γ_∞ has finite index in the stabilizer $\mathrm{Stab}_{\Gamma_G}(\Omega_Y)$ of Ω_Y in Γ_G. Thus, there is an element γ in $\mathrm{Stab}_{\Gamma_G}(\Omega_Y)$ that does not lie in Γ_∞ but so that $\gamma^k \in \Gamma_\infty$.

By a standard argument [JM, Lem. 3.6] if ρ_i are representations determined by (Q_i, ω_i), and we have $\rho_i(g) \to \gamma^k$ for some $g \in \pi_1(S)$, and $\rho_i(h_i) \to \gamma$ for $h_i \in \pi_1(S)$, then $g = h_i^k$ for all $i \gg 0$. By unique divisibility of $\pi_1(S)$, $h_i = h \in \pi_1(S)$ for all $i \gg 0$ and thus $\rho_i(h) \to \gamma$. Thus, γ lies in Γ_∞, so $\overline{\pi}$ is an embedding. □

Iteration converges. We now apply the gluing lemma to the setting of iteration to prove lemma 6.2. Theorem 6.1 will follow.

Proof (of lemma 6.2). Fix $(X, Y) \in \mathrm{Teich}(S) \times \mathrm{Teich}(\overline{S})$. Let $Q_i = Q(\varphi^i X, Y)$ denote the iteration of the mapping class φ on the Bers slice B_Y. The proof divides into cases.

Case 1: $S_{\mathsf{F}}(\varphi)$ is the entire surface S. If $S_{\mathsf{F}}(\varphi) = S$ then φ has finite order s. Then the s algebraic and geometric accumulation points are

$$Q(X, Y),\ Q(\varphi X, Y), \dots, Q(\varphi^{s-1} X, Y),$$

all of which are quasi-Fuchsian, and thence mutually quasi-isometric.

Case 2: $S_{\mathsf{F}}(\varphi)$ is empty. Decomposition 4.8 guarantees that the limit Q_φ has a unique component $Y = \partial\overline{Q_\varphi}$ in its conformal boundary if and only if $S_{\mathsf{F}}(\varphi) = \varnothing$. Hence Q_φ is totally degenerate.

We claim any such accumulation point is a strong limit: i.e. $N = Q_\varphi$. Let $Q_\varphi = \mathbb{H}^3/\Gamma_\varphi$ cover $N = \mathbb{H}^3/\Gamma_G$ so that $\Gamma_\varphi < \Gamma_G$. By corollary 6.7, the unique component Ω_Y of the domain of discontinuity $\Omega(\Gamma_\varphi)$ embeds in the domain of discontinuity $\Omega(\Gamma_G)$, so $\Omega(\Gamma_G) = \Omega_Y$. Moreover, the natural covering $\pi\colon Q_\varphi \to N$ extends to a holomorphic embedding on the unique component $Y \subset \partial\overline{Q_\varphi}$. Thus the covering

$$\Omega_Y/\Gamma_\varphi \to \Omega(\Gamma_G)/\Gamma_G$$

is a homeomorphism and we have $\Gamma_\varphi = \Gamma_G$. It follows that $N = Q_\varphi$, and likewise $N' = Q'_\varphi$ so this case follows from theorem 5.7.

Case 3: $S_{\mathsf{F}}(\varphi)$ is neither the entire surface S, nor empty. We reduce to the case that φ is stable as follows.

Reduction to stable case. Let $Q_i = Q(\varphi^i X, Y)$. As remarked in section 4, if a subsequence Q_{i_j} converges to Q_φ then for any accumulation point Q'_φ of Q_{i_j+1}, we have $d(Q_\varphi, Q'_\varphi) < \infty$ by lower semi-continuity of $d(.,.)$ [Mc2, Prop. 3.1]. In other words, there is a marking preserving quasi-isometry from Q_φ to Q'_φ.

A similar argument works for geometric limits: pass to a subsequence so that $\{Q_{i_j}\}$ converges geometrically to a limit N covered by Q_φ. Let Q'_φ be a limit of $\{Q_{i_j+1}\}$ covering N' after passing to further subsequences. Then as in the proof of theorem 5.4, quasi-isometries $\Theta_j\colon Q_{i_j} \to Q_{i_j+1}$ have lifts that extend to uniformly quasiconformal conjugacies, which converge up to subsequence. Any limit descends to a marking-preserving quasi-isometry $\widetilde{\Theta}\colon Q_\varphi \to Q'_\varphi$ that covers a quasi-isometry $\Theta\colon N \to N'$ between pairs of geometric accumulation points.

Thus, it suffices to prove lemma 6.2 for any finite power of φ, so we may assume φ is stable. We fix our attention on one component $F_0 \subset S_{\mathsf{F}}(\varphi)$. The stable mapping class φ stabilizes the isotopy class of each element $\eta \in \pi_1(F_0)$.

The re-marking trick. The re-markings $\varphi^{-i}(Q(\varphi^i X, Y)) = Q(X, \varphi^{-i} Y)$ range in a precompact slice of $QF(S)$ obtained by fixing the first factor in the parameterization. After passing to successive subsequences so that $\{Q(\varphi^i X, Y)\}$ converges algebraically to Q_φ, and geometrically to a limit N covered by Q_φ, pass to a further algebraically convergent sequence of re-markings $\{Q(X, \varphi^{-i} Y)\}$ converging to $Q_{\varphi^{-1}}$. Let

$$(f_i\colon S \to Q_i) \to (f\colon S \to Q_\varphi) \quad \text{and} \quad (g_i\colon S \to Q_i) \to (g\colon S \to Q_{\varphi^{-1}})$$

denote their implicit markings. By theorem 4.7, for each component F_0 of $S_{\mathsf{F}}(\varphi)$ the subgroup $f_*(\pi_1(F_0))$ is quasi-Fuchsian, as is $g_*(\pi_1(F_0))$.

Let

$$\partial_{\mathrm{qf}} Q_\varphi = W_1 \sqcup \ldots \sqcup W_q \quad \text{and} \quad \partial_{\mathrm{qf}} Q_{\varphi^{-1}} = Z_1 \sqcup \ldots \sqcup Z_q$$

be quasi-Fuchsian conformal boundary components such that W_n is uniformized by $f_*(\pi_1(F_n))$ and Z_n is uniformized by $g_*(\pi_1(F_n))$ for each $F_n \subset S_{\mathsf{F}}(\varphi)$. There is a natural gluing involution

$$\tau\colon \partial_{\mathrm{qf}} Q_\varphi \rightleftharpoons \partial_{\mathrm{qf}} Q_{\varphi^{-1}}$$

determined up to isotopy by the condition that $\tau_* = id_{\pi_1(F_n)}$ for each $n = 1, \ldots, q$.

Lemma 6.8 THE RE-MARKING LEMMA. *The manifold $Q_\varphi \sqcup Q_{\varphi^{-1}}$ solves the gluing problem τ, and $(Q_\varphi \sqcup Q_{\varphi^{-1}})/\tau$ is isometric to the geometric limit N.*

Proof. Using the same indices, we pass to the chosen subsequences of

$$Q(\varphi^i X, Y) = (f_i\colon S \to Q_i) \quad \text{and} \quad Q(X, \varphi^{-i} Y) = (g_i\colon S \to Q_i).$$

We apply the gluing lemma to these subsequences with respect to the surface $F_0 \subset S_{\mathsf{F}}(\varphi)$. We claim these subsequences satisfy its hypotheses.

First, since φ preserves the isotopy class of each $\eta \in \pi_1(F_0)$, the subgroup $(f_i)_*|_{\pi_1(F_0)}$ is conjugate to $(g_i)_*|_{\pi_1(F_0)}$ in $\pi_1(Q_i)$. So $Q_{\varphi^{-1}}$ also covers N by a locally isometric covering compatible with f_* and g_* on $\pi_1(F_0)$.

We now check the other hypotheses.

Hypothesis a): That the restriction $f_*(\pi_1(F_0))$ of f_* to $\pi_1(F_0)$ has quasi-Fuchsian image (and similarly for g_*) follows from theorem 4.7. The coverings of Q_φ and $Q_{\varphi^{-1}}$ corresponding to $f_*(\pi_1(F_0))$ and $g_*(\pi_1(F_0))$ are isometrically identified with the quasi-Fuchsian manifold $Q_0 \in QF(F_0)$.

Hypothesis b): It remains to verify that in passing to the quasi-Fuchsian cover Q_0 the surfaces $W_0 \subset \partial_{\mathrm{qf}} Q_\varphi$ and $Z_0 \subset \partial_{\mathrm{qf}} Q_{\varphi^{-1}}$ uniformized by $f_*(\pi_1(F_0))$ and $g_*(\pi_1(F_0))$ lift to distinct components of ∂Q_0. We appeal to baseframes and

Kleinian groups to make use of Carathéodory convergence. Fix a baseframe $\omega \in Q_0$. Then (Q_0, ω) determines the quasi-Fuchsian group Γ_0. Let

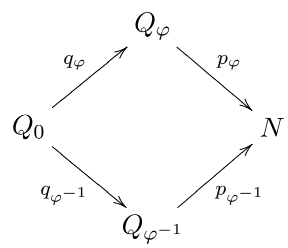
$$q_\varphi \colon Q_0 \to Q_\varphi \quad \text{and} \quad q_{\varphi^{-1}} \colon Q_0 \to Q_{\varphi^{-1}}$$

denote the locally isometric covering projections, and let

$$(Q_\varphi, q_\varphi(\omega)) = (\mathbb{H}^3, \widetilde{\omega})/\Gamma_\varphi \quad \text{and} \quad (Q_{\varphi^{-1}}, q_{\varphi^{-1}}(\omega)) = (\mathbb{H}^3, \widetilde{\omega})/\Gamma_{\varphi^{-1}}$$

for Kleinian groups Γ_φ and $\Gamma_{\varphi^{-1}}$. Then Γ_0 is a subgroup of Γ_φ and $\Gamma_{\varphi^{-1}}$.

The coverings $p_\varphi \colon Q_\varphi \to N$ and $p_{\varphi^{-1}} \colon Q_{\varphi^{-1}} \to N$, being compatible on $\pi_1(F_0)$, determine the diagram of covering spaces



The baseframe $p_\varphi \circ q_\varphi(\omega) \in N$ determines the geometric limit Kleinian group Γ_G so that $\Gamma_\varphi < \Gamma_G$ and $\Gamma_{\varphi^{-1}} < \Gamma_G$.

Let $\Omega_Y \subset \Omega(\Gamma_\varphi)$ and $\Omega_X \subset \Omega(\Gamma_{\varphi^{-1}})$ denote the invariant components for Γ_φ and $\Gamma_{\varphi^{-1}}$. The components Ω_Y and Ω_X cover $Y = \Omega_Y/\Gamma_\varphi$ and $X = \Omega_X/\Gamma_{\varphi^{-1}}$. The limit sets are their frontiers $\Lambda(\Gamma_\varphi) = \partial\Omega_Y$ and $\Lambda(\Gamma_{\varphi^{-1}}) = \partial\Omega_X$.

By Carathéodory convergence (proposition 6.6), the domains Ω_X and Ω_Y embed disjointly in the domain of discontinuity $\Omega(\Gamma_G)$ so Ω_X lies in some complementary component $D \subset \widehat{\mathbb{C}} - \Omega_Y$. Since D covers a quasi-Fuchsian component of ∂Q_φ, ∂D is a Jordan curve. As Γ_0 is a common subgroup of Γ_φ and $\Gamma_{\varphi^{-1}}$, its limit set $\Lambda(\Gamma_0)$ lies in the intersection $\partial\Omega_X \cap \partial\Omega_Y$ which is contained in ∂D. Since $\Lambda(\Gamma_0)$ is a Jordan curve, $\Lambda(\Gamma_0) = \partial D = \partial\Omega_Y \cap \partial\Omega_X$. Thus Ω_X and Ω_Y lie on different sides of $\Lambda(\Gamma_0)$, and we conclude that the components $W_0 \subset \partial Q_\varphi$ and $Z_0 \subset \partial Q_{\varphi^{-1}}$ lift to distinct components of ∂Q_0.

Having satisfied the hypotheses of lemma 6.5 for arbitrary $F_0 \subset S_{\mathsf{F}}(\varphi)$, the manifold $Q_\varphi \sqcup Q_{\varphi^{-1}}$ is a solution to the gluing problem $\tau \colon \partial_{\mathrm{qf}} Q_\varphi \rightleftharpoons \partial_{\mathrm{qf}} Q_{\varphi^{-1}}$ that covers the geometric limit N compatibly with τ. Applying **GIII**, $(Q_\varphi \sqcup Q_{\varphi^{-1}})/\tau$ covers N by a local isometry

$$\pi \colon (Q_\varphi \sqcup Q_{\varphi^{-1}})/\tau \to N.$$

Applying **GI**, we have $\partial\left((Q_\varphi \sqcup Q_{\varphi^{-1}})/\tau\right) = X \sqcup Y$. Letting $\Gamma_\tau < \Gamma_G$ be the subgroup so that $(Q_\varphi \sqcup Q_{\varphi^{-1}})/\tau = \mathbb{H}^3/\Gamma_\tau$, the domain of discontinuity $\Omega(\Gamma_\tau)$ consists of Ω_X, Ω_Y and their translates under Γ_τ. Since Γ_τ is a subgroup of Γ_G, we have $\Omega(\Gamma_G) \subset \Omega(\Gamma_\tau)$. By proposition 6.6, the domains Ω_X and Ω_Y embed in Ω_G, so since each element $\gamma \in \Gamma_\tau$ lies in Γ_G, all translates of Ω_X and Ω_Y by elements of Γ_τ also embed in $\Omega(\Gamma_G)$. Thus we have $\Omega(\Gamma_\tau) \subset \Omega(\Gamma_G)$ which implies that

$$\Omega(\Gamma_\tau) = \Omega(\Gamma_G).$$

It follows that the locally isometric covering π extends to a covering map

$$\overline{\pi}\colon \Omega(\Gamma_G)/\Gamma_\tau \to \Omega(\Gamma_G)/\Gamma_G.$$

But by corollary 6.7, $\overline{\pi}$ is an embedding on each component X and Y of

$$\partial\left((Q_\varphi \sqcup Q_{\varphi^{-1}})/\tau\right).$$

By proposition 6.6, the orbits $\Gamma_G(\Omega_X)$ and $\Gamma_G(\Omega_Y)$ are disjoint, so π extends to an embedding $\overline{\pi}$ on the disjoint union $X \sqcup Y$. Since $X \sqcup Y = \Omega(\Gamma_G)/\Gamma_\tau$, it follows that $\overline{\pi}$ is a holomorphic isomorphism, so $\Gamma_\tau = \Gamma_G$. Thus, π is an isometry. □

Remark: In the final step of the proof, use of the conformal boundary obviates the need for any discussion of how simply degenerate ends cover in the natural covering $(Q_\varphi \sqcup Q_{\varphi^{-1}})/\tau \to N$, a topic of considerable subtlety and interest in its own right (see [Th1, §9] [Can2] [AC], for example).

Continuation of the proof of lemma 6.2: By theorem 5.7, there are marking-preserving quasi-isometries

$$\Theta_\varphi\colon Q_\varphi \to Q'_\varphi \quad \text{and} \quad \Theta_{\varphi^{-1}}\colon Q_{\varphi^{-1}} \to Q'_{\varphi^{-1}}$$

which we view as a marking-preserving quasi-isometry

$$\Theta\colon Q_\varphi \sqcup Q_{\varphi^{-1}} \to Q'_\varphi \sqcup Q'_{\varphi^{-1}}.$$

The disjoint unions $Q_\varphi \sqcup Q_{\varphi^{-1}}$ and $Q'_\varphi \sqcup Q'_{\varphi^{-1}}$, then, lie in the same deformation space, and they are solutions to a common gluing problem

$$\tau\colon \partial_{\mathrm{qf}} Q_\varphi \rightleftharpoons \partial_{\mathrm{qf}} Q_{\varphi^{-1}},$$

namely, the natural gluing problem of the re-marking lemma (lemma 6.8). By theorem 6.4, there is a quasi-isometry

$$\Theta_\tau\colon (Q_\varphi \sqcup Q_{\varphi^{-1}})/\tau \to (Q'_\varphi \sqcup Q'_{\varphi^{-1}})/\tau.$$

By the re-marking lemma, Θ_τ is a quasi-isometry

$$\Theta_\tau\colon N \to N'$$

between the geometric accumulation points N, and N' of the iteration of φ on B_Y. Lemma 6.2 follows, since Θ_τ is covered by a marking-preserving quasi-isometry $\widetilde{\Theta_\tau}\colon Q_\varphi \to Q'_\varphi$. □

Remark: The quasi-isometry Θ_τ also lifts to a marking-preserving quasi-isometry

$$\widetilde{\Theta_\tau}'\colon Q_{\varphi^{-1}} \to Q'_{\varphi^{-1}}.$$

Applying lemma 6.2, we prove geometric, and thence algebraic, convergence.

Proof (of theorem 6.1). Consider again the cases of lemma 6.2.

Case 1: If $S_{\mathsf{F}}(\varphi) = S$ then stability implies $\varphi = id$ and there is nothing to prove.

Case 2: If $S_{\mathsf{F}}(\varphi) = \varnothing$, then for any pair of accumulation points Q_φ and Q'_φ of $\{Q(\varphi^i X, Y)\}$, Case 2 of the proof of lemma 6.2 implies that $\partial Q_\varphi = Y = \partial Q'_\varphi$ and that Q_φ and Q'_φ are each strong limits. Since

$$\text{Def}(Q_\varphi) = \text{Teich}(\partial Q_\varphi) = \text{Teich}(\overline{S}),$$

and there is a marking-preserving quasi-isometry $\Theta \colon Q_\varphi \to Q'_\varphi$, Q_φ and Q'_φ represent the same point in Def(Q_φ). Hence $Q_\varphi = Q'_\varphi$, and the sequence $\{Q(\varphi^i X, Y)\}$ converges strongly to Q_φ.

Case 3: Assume $S_{\mathsf{F}}(\varphi) \neq S, \varnothing$. Let Q_φ and Q'_φ be any two algebraic accumulation points of $\{Q(\varphi^i X, Y)\}_{i=1}^\infty$ and let N and N' be geometric accumulation points they cover. As in the re-marking lemma, let $Q_{\varphi^{-1}}$ and $Q'_{\varphi^{-1}}$ be corresponding limits of the re-markings $\{Q(X, \varphi^{-i} Y)\}$ after passing to further subsequences.

By lemma 6.2 (and its proof), there is a quasi-isometry

$$\Theta_\tau \colon N \to N'$$

that lifts to quasi-isometries $\widetilde{\Theta_\tau} \colon Q_\varphi \to Q'_\varphi$ and $\widetilde{\Theta_\tau}' \colon Q_{\varphi^{-1}} \to Q'_{\varphi^{-1}}$ each compatible with markings.

Since the injectivity radius of N is bounded throughout its convex core and its conformal boundary is incompressible, we have

$$\text{Def}(N) = \text{Teich}(\partial N) = \text{Teich}(S) \times \text{Teich}(\overline{S}).$$

As $\partial N = X \sqcup Y = \partial N'$, Θ_τ is homotopic to an isometry $\xi \colon N \to N'$. Geometric convergence follows.

Lifting ξ to a marking-preserving isometry $\widetilde{\xi} \colon Q_\varphi \to Q'_\varphi$, we conclude that $\{Q(\varphi^i X, Y)\}$ converges algebraically as well. □

Corollary 6.9 STRONG CONVERGENCE. *Let* $\varphi \in \text{Mod}(S)$ *be a mapping class. Then the sequence* $\{Q(\varphi^i X, Y)\}$ *converges strongly if and only if the finite order subsurface* $S_{\mathsf{F}}(\varphi)$ *is empty or* $S_{\mathsf{F}}(\varphi) = S$.

If $S_{\mathsf{F}}(\varphi) = S$ *then the limit does not depend on* X.

Proof. We proved the condition is sufficient above. To see it is also necessary, observe that when $S_{\mathsf{F}}(\varphi)$ is not the whole surface and also non-empty, lemma 6.8 realizes any geometric limit N as a nontrivial gluing $(Q_\varphi \sqcup Q_{\varphi^{-1}})/\tau$ of algebraic accumulation points Q_φ and $Q_{\varphi^{-1}}$. It follows that the natural covering map $Q_\varphi \to N$ is not an embedding, and the convergence is not strong.

Replacing X with X' produces a limit Q'_φ lying in Def(Q_φ). When $S_{\mathsf{F}}(\varphi) = S$ we again have $\partial Q_\varphi = Y = \partial Q'_\varphi$, so $Q_\varphi = Q'_\varphi$ and the limit does not depend on X. □

For completeness, we include the following corollary (cf. [Bers4, Lem. 2a]). An essential subsurface $S' \subset S$ is *rigid* if int(S') consists of triply punctured spheres.

Corollary 6.10 *Let $\varphi \in \mathrm{Mod}(S)$ be a mapping class. If the finite order subsurface $S_{\mathsf{F}}(\varphi)$ is rigid, then for any $(X,Y) \in \mathrm{Teich}(S) \times \mathrm{Teich}(\overline{S})$, the sequence $\{Q(\varphi^i X, Y)\}_{i=1}^{\infty}$ converges algebraically to a limit that does not depend on X.*

Proof. If $S_{\mathsf{F}}(\varphi)$ is rigid, then for any limit Q_φ of $\{Q(\varphi^i X, Y)\}$ the surfaces in ∂Q_φ other than Y have no moduli. Thus, $\mathrm{Def}(Q_\varphi) = \mathrm{Teich}(\overline{S})$. The same holds for any limit Q'_φ of $\{Q(\varphi^i X', Y)\}$ with X' in place of X, so Q_φ and Q'_φ determine the same point in $\mathrm{Def}(Q_\varphi)$. □

Corollary 6.11 *If $\dim_{\mathbb{C}}(\mathrm{Teich}(S)) = 1$, and $\varphi \in \mathrm{Mod}(S)$ has infinite order, then for any $(X,Y) \in \mathrm{Teich}(S) \times \mathrm{Teich}(\overline{S})$ the sequence $\{Q(\varphi^i X, Y)\}_{i=1}^{\infty}$ converges algebraically to a limit that does not depend on X.*

Proof. Since the dimension $\dim_{\mathbb{C}}(\mathrm{Teich}(S)) = 1$, either $S_{\mathsf{F}}(\varphi) = \varnothing$ or $\mathrm{int}(S_{\mathsf{F}}(\varphi))$ is a pair of triply punctured spheres. Hence $S_{\mathsf{F}}(\varphi)$ is either empty or rigid, and the corollary follows from corollaries 6.9 and 6.10. □

Finite order behavior. Finally, we justify the hypothesis of stability.

Theorem 6.12 FINITE-ORDER NON-CONVERGENCE. *Given any S for which the complex dimension $\dim_{\mathbb{C}}(\mathrm{Teich}(S)) > 1$, there exists $\varphi \in \mathrm{Mod}(S)$ of infinite order and $X \in \mathrm{Teich}(S)$ such that in any Bers slice B_Y the sequence $\{Q(\varphi^i X, Y)\}_{i=1}^{\infty}$ has more than one algebraic accumulation point in $\overline{B_Y}$.*

Proof. Let S have genus g with n boundary components. The complex dimension of the Teichmüller space $\mathrm{Teich}(S)$ is given in terms of g and n by the formula

$$\dim_{\mathbb{C}}(\mathrm{Teich}(S)) = 3g - 3 + n.$$

If $\dim_{\mathbb{C}}(\mathrm{Teich}(S)) > 1$ then either $g = 0$ and $n \geq 5$, $g = 1$ and $n \geq 2$ or $g \geq 2$. There is an essential proper subsurface $T \subset S$ such that the interior $\mathrm{int}(T)$ is homeomorphic either to a punctured torus or a sphere with four points removed, and so that T has precisely one boundary component $\gamma \subset \partial T$ not in common with S (see figure 8). Then there is a mapping class $\varphi \in \mathrm{Mod}(S)$ so that

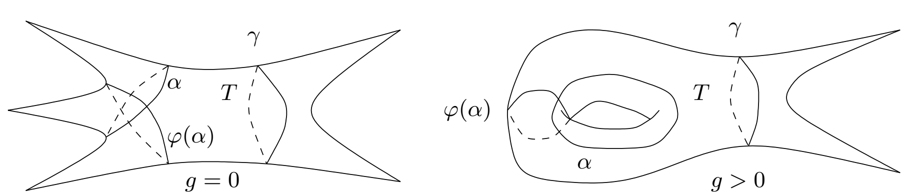


Figure 8. The subsurface $T \subset S$.

1. φ induces an order s element of $\mathrm{Mod}(T)$ via restriction,

2. there is an isotopy class of essential non-peripheral simple closed curves $\alpha \subset T$ with $i(\varphi(\alpha), \alpha) \neq 0$, and
3. φ^s is a single Dehn twist about γ.

Since φ^s is stable, corollary 6.3 implies that for any $X \in \text{Teich}(S)$ the sequence $\{Q(\varphi^i X, Y)\}_{i=1}^\infty$ has a most s accumulation points

$$Q_a = \lim_{i \to \infty} Q(\varphi^{si+a} X, Y), \quad a = 0, \dots, s-1,$$

with geometric limits $N_0, \dots, N_{s-1}$.

The geometric limit N_0 has conformal boundary $\partial \overline{N_0} = X \sqcup Y$ in the markings induced by the gluing in lemma 6.2. Likewise, we have $\partial \overline{N_1} = \varphi X \sqcup Y$ since this limit is obtained from iteration of φ^s on B_Y beginning at $Q(\varphi X, Y)$.

The algebraic limits Q_0 and Q_1 each have quasi-Fuchsian covers corresponding to T. Let $Z_0 \subset \partial \overline{Q_0}$ and $Z_1 \subset \partial \overline{Q_1}$ be elements of $\text{Teich}(T)$ in the conformal boundary of Q_0 and Q_1 uniformized by the quasi-Fuchsian subgroup corresponding to T. In the covering $Q_0 \to N_0$, the surface $X \subset \partial \overline{N_0}$ lifts to a surface $\widetilde{X} \subset \partial \overline{Q_0}$ that includes into the surface Z_0. Likewise $\varphi X \subset \partial \overline{N_1}$ lifts to a surface $\widetilde{\varphi X} \hookrightarrow Z_1$.

By the collar lemma (see [Bus, Thm. 4.1.1]), given $K > 0$ we may choose $\epsilon > 0$ so that if X is a Riemann surface on which the simple closed curve α has $\text{length}_X(\alpha) < \epsilon$ then the annular K-neighborhood of $\widetilde{\alpha}$ in the cover $\widetilde{X}$ of X corresponding to $\langle \alpha \rangle \cong \mathbb{Z}$ embeds in the covering projection to X.

Taking K sufficiently large to ensure that the corresponding ϵ is less than K, we choose $X \in \text{Teich}(S)$ on which $\text{length}_X(\alpha) < \epsilon$. Then

$$\text{length}_{\varphi X}(\varphi(\alpha)) < \epsilon \quad \text{and} \quad \text{length}_{\varphi X}(\alpha) > K.$$

The coverings $\widetilde{X} \to X$ and $\widetilde{\varphi X} \to \varphi X$ are isometric in the Poincaré metrics, and the inclusions $\widetilde{X} \hookrightarrow Z_0$ and $\widetilde{\varphi X} \hookrightarrow Z_1$ induce contractions of the Poincaré metrics [Mc2, Prop. 4.9]. It follows that

$$\text{length}_{Z_0}(\alpha) < \epsilon \quad \text{and} \quad \text{length}_{Z_1}(\varphi(\alpha)) < \epsilon.$$

Applying the collar lemma, we have that $\text{length}_{Z_1}(\alpha) > 2K > \epsilon$, which implies that $Z_0 \neq Z_1$ in $\text{Teich}(T)$.

Hence, there is no isometry between Q_0 and Q_1 that respects markings, so Q_0 and Q_1 represent distinct points in ∂B_Y. $\square$

7 Quasi-isometric models

In this section we describe how quasi-isometric models for the algebraic and geometric limits of iteration of $\varphi \in \text{Mod}(S)$ on a Bers' slice can be constructed directly from φ.

Models for algebraic limits. It suffices to build a quasi-isometric model for limits of stable iteration. Let $\varphi \in \text{Mod}(S)$ be a stable mapping class.

Structure of Q_φ. Let P_φ be the cuspidal thin part of Q_φ. Let $\{A_j\}_{j=1}^k$ enumerate the annuli in $\overline{\mathcal{N}(\Pi(\varphi))}$, let $\{S_m\}_{m=1}^p$ enumerate components of $S_{\mathsf{P}}(\varphi)$, and let $\{F_n\}_{n=1}^q$ enumerate components of $S_{\mathsf{F}}(\varphi)$.

Then as in decomposition 4.8, any relative compact core

$$\iota_\varphi \colon (\mathcal{M}_\varphi, \mathcal{P}_\varphi) \to (Q_\varphi - P_\varphi, \partial P_\varphi)$$

has the following form up to diffeomorphism.

1. $\mathcal{M}_\varphi \cong S \times I$,
2. $\mathcal{P}_\varphi = (\partial S \times I) \cup (\cup_{j=1}^k A_j \times \{0\})$, and
3. $\partial_0 \mathcal{M}_\varphi = Y \cup (\cup_{m=1}^p S_m \times \{0\}) \cup (\cup_{n=1}^q F_n \times \{0\})$ where $Y = S \times 1$.

For any mapping class $\psi \in \mathrm{Mod}(S)$, the mapping torus T_ψ is quasi-isometrically unique by compactness of its pared submanifold. Thus, the cover M_ψ corresponding to the fiber is as well, so when ψ is pseudo-Anosov we may describe the quasi-isometric geometry of M_ψ without reference to specific hyperbolic structures.

To construct a quasi-isometric model for Q_φ, we note that by corollary 2.14 it suffices to give a quasi-isometric model for the pared submanifold of its convex core; the geometrically finite ends carry no essential quasi-isometric data.

We take any differentiable structure on $(\mathcal{M}_\varphi, \mathcal{P}_\varphi)$ above.

Periodic ends: Let ψ_m be the pseudo-Anosov mapping class induced on S_m by the first iterate of φ that leaves S_m invariant. For each surface $S_m \times \{0\}$ *construct* an end $\mathcal{E}_m^1$ by fixing a smooth structure on $(S_m \times \{0\}) \times [0,1]$ and gluing successive copies end to end in the negative direction by a diffeomorphism representing ψ_m. The resulting end $\mathcal{E}_m^1$, marked by the inclusion $\iota_m \colon S_m \to \mathcal{E}_m^1$ is quasi-isometric to the negative end of $M_{\psi_m^{-1}}$ marked by the lift of the fiber S_m.

Adjoin each $\mathcal{E}_m^1$, $m = 1, \dots, p$, to $(\mathcal{M}_\varphi, \mathcal{P}_\varphi)$ along $S_m \times \{0\}$ by the identity. Call the resulting manifold $\mathfrak{M}_\varphi$. Then we have the following.

Proposition 7.1 *The model $\mathfrak{M}\varphi$ is quasi-isometric to a neighborhood of the pared submanifold of the convex core of Q_φ.* □

Models for geometric limits. When iteration is not strongly convergent, N is realized as the gluing $(Q_\varphi \sqcup Q_{\varphi^{-1}})/\tau$ of Q_φ with the limit $Q_{\varphi^{-1}}$ of its re-markings $\{Q(X, \varphi^{-i} Y)\}_{i=1}^\infty$, by lemma 6.8. In this case, the Klein-Maskit theory (see theorem 6.4) gives a quasi-isometric model for any gluing M/τ explicitly in terms of M and the gluing data. From this argument, it follows that the geometric limit N has a standard quasi-isometric model when the iteration is not strongly convergent.

Structure of $Q_{\varphi^{-1}}$. As above, let $P_{\varphi^{-1}}$ be the cuspidal thin part of $Q_{\varphi^{-1}}$. Then $Q_{\varphi^{-1}}$ too has a relative compact core

$$\iota_{\varphi^{-1}} \colon (\mathcal{M}_{\varphi^{-1}}, \mathcal{P}_{\varphi^{-1}}) \to (Q_{\varphi^{-1}} - P_{\varphi^{-1}}, \partial P_{\varphi^{-1}})$$

with the following form up to diffeomorphism.

1. $\mathcal{M}_{\varphi^{-1}} \cong S \times I$,

2. $\mathcal{P}_{\varphi^{-1}} = (\partial S \times I) \cup (\cup_{j=1}^{k} A_j \times \{1\})$, and
3. $\partial_0 \mathcal{M}_{\varphi^{-1}} = X \cup (\cup_{m=1}^{p} S_m \times \{1\}) \cup (\cup_{n=1}^{q} F_n \times \{1\})$ where $X = S \times \{0\}$.

As above, for each surface S_m construct an end $\mathcal{E}'_m$ by gluing a half infinite collection of copies of $S_m \times \{1\} \times [0,1]$ end to end in the *positive* direction by a diffeomorphism representing ψ_m^{-1}. Then $\mathcal{E}'_m$ is quasi-isometric to the *positive* end of the pared submanifold of $M_{\psi_m^{-1}}$. After adjoining each end $\mathcal{E}'_m$ to $(\mathcal{M}_{\varphi^{-1}}, \mathcal{P}_{\varphi^{-1}})$ along $S_m \times \{1\}$, denote the resulting model for the pared submanifold of the convex core by $\mathfrak{M}_{\varphi^{-1}}$.

Gluing: Let $T\colon \{\cup_{n=1}^{q} F_n \times \{0\}\} \to \{\cup_{n=1}^{q} F_n \times \{1\}\}$ be the defined by the identification $T(x,0) = (x,1)$. Then by the proof of theorem 6.4, and lemma 6.8 this gluing determines the quasi-isometric model for the geometric limit (figure 9).

Theorem 7.2 *If $Q(\varphi^i X, Y)$ does not converge strongly, then the gluing*

$$\mathfrak{M}_{\varphi} \bigcup_{T} \mathfrak{M}_{\varphi^{-1}}$$

is quasi-isometric to a smooth neighborhood of the pared submanifold of the convex core of N. $\square$

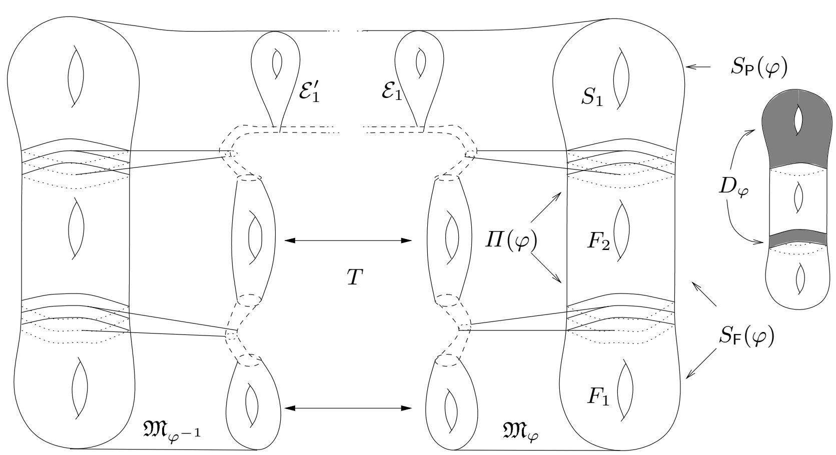


Figure 9. Gluing the models $\mathfrak{M}_\varphi$ and $\mathfrak{M}_{\varphi^{-1}}$ by T.

Recall that $D_\varphi \subset S$ is the subsurface of S determined up to isotopy by $S - S_{\mathsf{F}}(\varphi)$. As a consequence, we have:

Theorem 7.3 HOMEOMORPHISM TYPES. *Let N be the geometric limit of iteration of $\varphi \in \mathrm{Mod}(S)$. Then either $D_\varphi = S$ and N has the homeomorphism type $N \cong \mathrm{int}(S) \times \mathbb{R}$, or N has the homeomorphism type*

$$N \cong \mathrm{int}(S) \times \mathbb{R} - D_\varphi \times \{0\}. \qquad \square$$

The annuli in D_φ recede to rank-2 cusps, while each of the subsurfaces $S_m \subset D_\varphi$ of negative Euler characteristic recede to infinity leaving a pair of quasi-periodic simply degenerate ends.

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