Coarse and synthetic Weil-Petersson geometry: quasi-flats, geodesics, and relative hyperbolicity

JEFFREY BROCK
HOWARD MASUR

We analyze the coarse geometry of the Weil-Petersson metric on Teichmüller space, focusing on applications to its synthetic geometry (in particular the behavior of geodesics). We settle the question of the strong relative hyperbolicity of the Weil-Petersson metric via consideration of its coarse quasi-isometric model, the *pants graph*. We show that in dimension 3 the pants graph is strongly relatively hyperbolic with respect to naturally defined product regions and show any quasi-flat lies a bounded distance from a single product. For all higher dimensions there is no non-trivial collection of subsets with respect to which it strongly relatively hyperbolic; this extends a theorem of [BDM] in dimension 6 and higher into the intermediate range (it is hyperbolic if and only if the dimension is 1 or 2 [BF]). Stability and relative stability of quasi-geodesics in dimensions up through 3 provide for a strong understanding of the behavior of geodesics and a complete description of the CAT(0)-boundary of the Weil-Petersson metric via curve-hierarchies and their associated *boundary laminations*.

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1 Introduction

The study of the large scale geometry of Teichmüller space and has given rise to new perspectives on Teichmüller geometry and dynamics in recent years, with results of Masur-Minsky [MM2], Brock [Br1], Brock-Farb [BF], Behrstock-Minsky [BM], Rafi [Raf], Wolpert [Wol2], and others giving insight into coarse phenomena that arise in consideration of various metrics. Notable among these is the Weil-Petersson metric on Teich(S), which carries a convenient coarse description in terms of combinatorics of pants decompositions of a surface (see [Br1]).

One feature that is common to virtually all these investigations is the emergence of obstructions to hyperbolicity (in the sense of Gromov) in higher dimensional cases – the Weil-Petersson metric is Gromov hyperbolic if and only if the Teichmüller space

has dimension at most 2 (see [BF]). When a space fails to be Gromov hyperbolic, the lack of stability properties familiar in negative curvature impedes an immediate understanding of the behavior of geodesics and quasi-geodesics.

Nevertheless, the notion of *strong relative hyperbolicity* allows for similar control of quasi-geodesics up to their behavior in certain well defined regions that are coarsely isolated from one another. In this paper we flesh out precisely in which cases the Weil-Petersson metric exhibits such strong relative hyperbolicity.

It is important to note that understanding the coarse, large scale structure of a metric space can lead to a precise understanding of fine structure in the setting of CAT(0) and NPC (non-positively curved) geometry. As an example, our methods provide for a complete description of the CAT(0) boundary of the Weil-Petersson metric up through Teichmüller spaces of dimension 3, originating out of a purely coarse combinatorial model.

Let $S = S_{g,n}$ be a compact surface of genus g with n boundary components. We define the *complexity* $\zeta(S)$ of S to be the integer $3g-3+n$, namely, the complex dimension of the corresponding Teichmüller space Teich(S), or the *Teichmüller dimension* of S. The initial focus of the paper will be on case $(g,n) \in \{(2,0),(1,3),(0,6)\}$. We say a curve γ is *domain separating* if $S \setminus \gamma$ has two components neither of which is a three-holed sphere.

A domain separating curve γ on S determines a set $X_\gamma \subset P(S)$ consisting of pants decompositions that contain γ. When $\zeta(S) = 3$ and γ is domain separating, the set X_γ naturally decomposes as a product of Farey-graphs, each naturally the pants graph on the complementary one-holed torus or four holed sphere in $S \setminus \gamma$. We show the following.

Theorem 1 *Let $S = S_{g,n}$ where $(g,n) \in \{(2,0),(1,3),(0,6)\}$. Then the pants graph $P(S)$ is strongly relatively hyperbolic relative to the sets X_γ where γ ranges over all domain separating curves in S.*

For the purposes of the proof, we will refer to the formulation of strong relative hyperbolicity given in [DS] and refined in [Dru].

Roughly speaking, relative hyperbolicity guarantees that by 'coning-off' each X_γ to a single point p_γ by edges of length one, the resulting metric is Gromov hyperbolic. The theorem asserts further that this relative hyperbolicity is *strong* in the sense that the subsets X_γ satisfy the bounded region (or coset) penetration property (cf. [Fa], [BF]).

In particular, this condition implies that when two quasi-geodesics in $P(S)$ begin and end at the same position, they enter and exit uniform neighborhoods of each X_γ within a bounded distance of one another. We give a more formal definition in section 3 and prove the requisite bounded coset penetration property in Proposition 1.

Our considerations have been motivated by the notion of a *hierarchy path*, a particular kind of quasi-geodesic arising out of the hyperbolicity of the curve complex. In [MM2], this transitive family of quasi-geodesics in $P(S)$ is described, built up from geodesics in the curve complexes of non-annular essential subsurfaces of S. Such a quasi-geodesic is called a *resolution* of a *hierarchy* $H(P_1, P_2)$ connecting P_1 and P_2 in $P(S)$.

In the above cases, given a resolution $\rho\colon [0,n] \to P(S)$ of a hierarchy $H(P_1,P_2)$ we denote by $X(\rho)$ the union of the image of ρ and the Farey-graph products $X_\gamma = P(W) \times P(W^c)$ where $\gamma \in \mathcal{C}(S)$ is the common boundary of W and W^c for which either W or W^c is a 'component domain' of $H(P_1,P_2)$ (see [MM2]).

Then Theorem 1 will follow from the following quasi-convexity result.

Theorem 2 *For any resolution ρ of the hierarchy $H(P_1,P_2)$, the union $X(\rho)$ is quasi-convex in $P(S)$.*

Quasi-convexity of this set guarantees that quasi-geodesics in $P(S)$, while not stable in the whole of $P(S)$, do satisfy a relative stability with respect to the product regions X_γ.

Rank and quasi-flats. A *quasi-flat* F in a metric space X is a quasi-isometric embedding[1]

$$F : \mathbb{R}^n \to X$$

where $n \geq 2$. The integer n is called the *rank* of the quasi-flat. The investigations due to [KL] and [EF] of quasi-isometric rigidity in the setting of higher rank symmetric spaces each used the classification of quasi-flats as a central tool.

Let $\zeta(S) = 3$, and let γ be a separating curve for which each complementary X_1 and X_2 satisfies $\zeta(X_i) = 1$. Then the Farey graph product $P(X_1) \times P(X_2)$ sits naturally in $P(S)$ as the subset $X_\gamma \subset P(S)$ consisting of pants decompositions containing the curve γ. Theorem 2 then allows one to give a classification of maximal quasiflats for $P(S)$ in the cases when $\zeta(S) = 3$.

[1]The map F is a *quasi-isometric embedding* if F distorts distances by a bounded additive and multiplicative amount.

Theorem 3 (QUASI-FLATS THEOREM) *Let $\zeta(S) = 3$. Then each quasi-flat F in $P(S)$ has rank 2 and lies a bounded distance from a product $P(X_1) \times P(X_2) \subset P(S)$ of Farey-graphs corresponding to the complementary subsurfaces of a domain-separating curve γ.*

The *geometric rank* of a metric space X is the maximal positive integer n for which X admits a quasi-isometric embedding $F\colon \mathbb{R}^n \to X$. As a consequence we obtain the following corollary, verifying a conjecture of the first author and Farb [BF] in the case $\zeta(S) = 3$. We remark that the following statement has been obtained independently (and for all surfaces S) by work of Behrstock and Minsky (see [BM]).

Corollary 4 (GEOMETRIC RANK) *When $\zeta(S) = 3$ the geometric rank of the pants graph $P(S)$ and hence the Weil-Petersson metric on* Teich(S) *is* 2.

The boundary of the Weil-Petersson metric. Our main theorem has applications for understanding the CAT(0) geometry of the Weil-Petersson metric.

The Weil-Petersson metric on the Teichmüller space Teich(S) has negative curvature, but it is not complete. Its completion $\overline{\text{Teich}(S)}$ has the structure of a CAT(0) space, namely, a geodesic metric space X in which pairs of points on edges of a geodesic triangle have distance at most that of the distance between corresponding points on a triangle in Euclidean space.

It is shown in [Br2] that the unit tangent spheres have no natural identification. However, the notion of an *asymptote class* of infinite geodesic rays is natural and basepoint independent. When X is Gromov hyperbolic, this CAT(0) boundary agrees with the usual Gromov boundary.

As a consequence of Theorem 2 and the main result of [BF], we give a description of the CAT(0) boundary of the Weil-Petersson metric when $3g - 3 + n \leq 3$.

We say λ is a *boundary lamination* if each component λ' of λ is a Gromov boundary point of $\mathcal{C}(S(\lambda'))$ where $S(\lambda')$ represents the minimal subsurface of S containing λ'. There is a natural topology on boundary laminations, which we formulate in section 4, but we warn the reader in advance that it is not continuous with respect to usual topologies on laminations arising out of consideration of transverse measures.

Theorem 5 *Let $S_{g,n}$ satisfy $3g - 3 + n \leq 3$. Then the* CAT(0) *boundary of the Weil-Petersson metric on* Teich(S) *is homeomorphic to the space of boundary laminations.*

We remark that both the notion of a boundary lamination defined here as well as the appropriate topology on the space of such depend on the strong characterization of quasi-geodesics of Theorem 2. Thus, with our methods, such a discussion is specific to dimension at most 3. In a separate paper with Minsky, we develop a general notion of a lamination associated to a Weil-Petersson geodesic ray that arises from convexity of length functions along geodesics (see [BMM]).

In the present discussion, the principal idea from Theorem 2 that infinite rays in the Weil-Petersson metric have associated "infinite hierarchies without annuli" in the sense of Masur and Minsky [MM2] gives rise to a natural way to associate infinite geodesics in the curve complex of S or its subsurfaces. Indeed, such the set of curves on S used to construct such 'hierarchies' has infinite diameter in the curve complex of some subsurface of Y of S. When there is a unique such Y, the hierarchy is essentially determined by the asymptotic data of a point in the Gromov-boundary $\partial\mathcal{C}(Y)$. In the case at hand, the only possibility if there is more than one such Y is that there are two such subsurfaces Y and Y^c, and the hierarchy in question has infinite diameter in each. In this case the related rate of divergence of the geodesic in each factor $\mathcal{C}(Y)$ and $\mathcal{C}(Y^c)$ determine an additional piece of data, the "slope" of the divergence of the ray. To encode this slope we associate real weights to the two laminations and projectivize. We will describe this in more detail in section 4.

Thickness and relative hyperbolicity. In the paper [BF] it was shown that for each S with $\zeta(S) \geq 3$ the pants graph $P(S)$, and thence the Weil-Petersson metric on Teich(S), is not Gromov hyperbolic. As the central obstruction to hyperbolicity is the existence of quasi-isometrically embedded product regions, one can ask whether a line of reasoning similar to the above approach to the case $\zeta(S) = 3$ might persist in higher complexity.

The paper [BDM] takes up this theme in generality; the notion of a *thick metric space* is introduced, and it is shown that for $\zeta(S) \geq 6$ the pants graph $P(S)$ is thick. This condition is equivalent ([BDM], [DS]) to the failure of strong relatively hyperbolicity in the sense of [DS].

The argument given in [BDM] for the thickness of the Weil-Petersson metric on Teichmüller space runs aground in the cases of mid-range complexity, namely $\zeta(S) = 4$ and 5. Pushing their approach a bit further we show that our strong relative hyperbolicity theorem for $\zeta(S) = 3$ is sharp in the following sense.

Theorem 6 *Let S be a surface with $\zeta(S) \geq 4$. Then pants graph $P(S)$ is not strongly relatively hyperbolic with respect to any co-infinite collection of subsets.*

Here, a subset $Y \subset X$ of a metric space Y is *co-infinite* if there are points in X at arbitrarily large distance from Y. The theorem is a consequence of the fact that in these cases the pants graph is *thick* in the sense of [BDM], shown in Theorem 18.

The theorem extends the relevant result of [BDM] which treats the case $\zeta(S) \geq 6$, to the intermediate range $\zeta(S) \in \{4,5\}$, and establishes the failure of strong relative hyperbolicity with respect to any collection of coinfinite subsets in general for $\zeta(S) \geq 4$.

Theorem 6 shows that the precise control of geodesics using these coarse methods stops in Teichmüller dimension 3 ($\zeta(S) = 3$). Interestingly, the central feature of these non-relatively-hyperbolic cases is the ability to "chain flats" in a gross sense: roughly speaking one can join any pair of points without ever leaving a union of quasi-isometrically embedded copies of $\mathbb{R}^2$. While this can be done as early as dimension 2 for the mapping class group, this kind of connectivity only begins in dimension 4.

Plan of the paper. We begin with preliminaries, condensing the notions required from the coarse geometry of the curve complex into the manageable formulation of a *hierarchy path*, namely, a particular type of quasi-geodesic in $P(S)$ arising inductively from the hyperbolicity of the curve complex. In section 3, we show that when $\zeta(S) = 3$ we have relative stability of quasi-geodesics in $P(S)$. In section 4, we deduce applications of this stability result to the finer structure of the CAT(0) boundary of the Weil-Petersson metric for S with $\zeta(S) \leq 3$. Finally, in section 5 we exhibit the "thickness" of the pants graph $P(S)$ and thence the Weil-Petersson metric for S with $\zeta(S) = 4$ and 5, and discuss how it follows naturally that the Weil-Petersson metric cannot be strongly relatively hyperbolic with respect to any collection of co-infinite subsets for $\zeta(S) \geq 4$. This final result shows that hyperbolicity and strong relative hyperbolicity, and the concomitant synthetic control, are sharp to Teichmüller dimension at most 3.

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2 Preliminaries

In this section we set out some of the preliminary notions we will need.

Complexes of curves and pants. Let S be a compact surface of negative Euler characteristic. The *curve complex* of S, denoted $\mathcal{C}(S)$, is the simplicial complex whose vertices correspond to isotopy classes of essential simple closed curves on S and whose k-simplices span collections of $k+1$ vertices whose corresponding simple closed curves can be realized pairwise-disjointly by simple closed curves on the surface. We will be primarily interested in the 1-skeleton of $\mathcal{C}(S)$, often called the *curve graph* of S, and the associated distance function $d_S(.,.)$ on the 0-skeleton induced by the metric obtained by assigning each edge length 1.

A related notion, the *pants graph* $P(S)$ associated to S is a graph whose vertices correspond to *pants decompositions*, namely, maximal families of distinct, essential, non-peripheral isotopy classes of simple closed curves on S so that the classes in the family have pairwise disjoint representatives.

In this case, edges connect vertices whose corresponding pants decompositions differ by an *elementary move*: pants decompositions P and P' differ by an elementary move if P' can be obtained from P by replacing one isotopy class α in P by another β so that representatives of β intersect representatives of α minimally among all possible choices for β. These moves have two types (see figure 1). In the first type $P \setminus \alpha$ contains a torus with a hole and β intersects α once. In the second $P \setminus \alpha$ contains a 4 holed sphere and β intersects α twice.

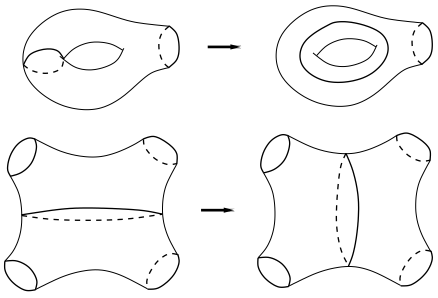

Figure 1. Elementary moves on Pants decompositions.

It is a theorem of Hatcher and Thurston that the graph $P(S)$ is connected (see [HT]), and thus there is a notion of distance between vertices obtained by metrizing $P(S)$ so that each edge has length 1. We denote by $d(\cdot,\cdot)$ this distance on $P(S)$.

Given a subsurface $Y \subseteq S$ that is not an annulus, work of Masur and Minsky [MM2] define a *coarse projection* π_Y from $\mathcal{C}(S)$ to uniformly bounded diameter subsets of $\mathcal{C}(Y)$, as follows.

If $\gamma \in \mathcal{C}(S)$ intersects Y essentially, we define $\pi_Y(\gamma)$ to be the collection of curves in $\mathcal{C}(Y)$ obtained by resolving the essential arcs of $\gamma \cap Y$ into simple closed curves. More precisely, if X is a finite area hyperbolic structure on $\text{int}(S)$ chosen for reference, one can consider the arcs of intersection of the geodesic representative γ^* of γ with the realization Y^* of Y as a subsurface of X with geodesic boundary. For each arc α of $\gamma^* \cap Y^*$, denote by α^* the shortest representative of α modulo the boundary of Y^* (i.e. up to homotopy with endpoints constrained to ∂Y^*). A regular neighborhood of $\alpha^* \cup \partial Y^*$ will have boundary consisting of simple closed curves in Y, some of which will be non-peripheral in Y. The union of these non-peripheral ones makes up the collection $\pi_Y(\gamma) \subset \mathcal{C}(Y)$. If this collection is empty, then $\pi_Y(\gamma)$ is defined to be the empty set.

If $\pi_Y(\gamma)$ is non-empty, it is easy to see that it has uniformly bounded diameter in $\mathcal{C}(Y)$. The projection π_Y is defined in exactly the same manner for pants decompositions P: resolve arcs of intersection of the geodesic representatives of P with the geodesic representative of Y into simple closed curves in Y and record those that are non-peripheral. Once again, it is easy to see that $\pi_Y(P)$ is a subset of $\mathcal{C}(Y)$ of uniformly bounded diameter.

We note for any non-annular subsurface Y, at least one curve in a pants decomposition P must have essential intersection with Y, so one always has $\pi_Y(P) \neq \emptyset$. For two pairs of pants P_1, P_2 we will use the notation $d_Y(P_1, P_2)$ to denote

$$\text{diam}_{\mathcal{C}(Y)}(\pi_Y(P_1) \cup \pi_Y(P_2)).$$

By a result of Masur and Minsky (see [MM1]), the curve complex $\mathcal{C}(Y)$ is δ-hyperbolic for some δ.

Let g_Y be a geodesic in $\mathcal{C}(Y)$.

Definition 7 *For any pants P decomposition, denote by $\pi_{g_Y}(P)$ a nearest point on the geodesic g_Y of the projection $\pi_Y(P)$.*

It follows from the hyperbolicity of $\mathcal{C}(Y)$ that this map is coarsely well-defined and that there is a constant $k \geq 1$ such that it is k-Lipschitz.

Definition 8 *The domain Y_1 is* nested *in the domain Y_2 if $Y_1 \subset Y_2$. The domains Y_1 and Y_2* intersect transversely *if they intersect and are not nested.*

Hierarchy paths. We will use a construction in [MM2] of a class of quasi-geodesics in $P(S)$ which we call *hierarchy paths* and which have the following properties. Given a positive function $f(x)$, and a real number $M > 0$, let

$$[f(x)]_M = \begin{cases} f(x) & \text{if } f(x) \geq M, \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

Definition 9 (HIERARCHY PATHS) *Any two pants decompositions P_1 and P_2 can be connected by at least one hierarchy path $\rho = \rho(P_1, P_2)\colon [0, n] \to P(S)$, with $\rho(0) = P_1$ and $\rho(n) = P_2$.*

These paths have the following properties.

(1) *There is a constant M_2 such that if Y is a subsurface of S with $\zeta(Y) \geq 1$ and $d_Y(P_1, P_2) \geq M_2$ then there is a maximal connected interval of times $I_Y = [t_1, t_2]$ such that for all $t \in I_Y$, ∂Y is a curve in $\rho(t)$. We will call such a subsurface Y a* component domain *of ρ. By convention, the full surface S is also a component domain.*

(2) *For each component domain Y, there is a geodesic $g_Y(s)$ in the curve complex $\mathcal{C}(Y)$, for s in a parameter interval J_Y, such that for each $t \in I_Y$, $\rho(t)$ contains a curve in $g_Y(s)$. Furthermore, the assignment $t \to s(t)$ is a monotonic function from I_Y to J_Y.*

(3) *If Y_1 and Y_2 are component domains that intersect transversely, then there is a notion $\prec_t$ of* time order *of the two domains which is the same for any hierarchy path joining P_1 and P_2. Time ordering has the property that there is a constant $M_1 \geq M_2$ such that if $Y_1 \prec_t Y_2$ then $d_{Y_2}(P_1, \partial Y_1) \leq M_1$ and $d_{Y_1}(P_2, \partial Y_2) \leq M_1$.*

(4) *For any component domain Y, if $I_Y = [t_1, t_2]$, then*

$$d_Y(\rho(t), \rho(t_1)) \leq M_1, \quad \text{if } t \leq t_1$$

and

$$d_Y(\rho(t), \rho(t_2)) \leq M_1, \quad \text{if } t \geq t_2.$$

(5) *There exist K', and C depending on M_1 so that*

$$d(P_1, P_2) \asymp_{K',C} \sum_{\substack{Y \subset S \\ Y \text{ non-annular}}} [d_Y(P_1, P_2)]_M, \tag{2.1}$$

where $\asymp_{K',C}$ denotes equality up to the multiplicative factor K' and additive constant C.

In [MM2] it is shown that $\prec_t$ defines a partial ordering on the component domains of a hierarchy path. The notion that two properly intersecting domains Y_1 and Y_2 are time-ordered refers to the fact that changes of projections to a pair of time ordered component domains do not occur simultaneously along a path but rather sequentially. We will denote by $\rho(P, P')$ a choice of a hierarchy path joining P to P' in $P(S)$.

Lemma 1 (TIME ORDER) *Suppose Y_1 and Y_2 are transversely intersecting domains, and P_1, P_2 are pants decompositions in $P(S)$ that satisfy*

$$d_{Y_1}(P_1, P_2) \geq 2M_1 \quad \text{and} \quad d_{Y_2}(P_1, P_2) \geq 2M_1.$$

Suppose $Y_1 \prec_t Y_2$ in $\rho(P_1, P_2)$. If R is another pants decomposition that satisfies

$$d_{Y_1}(R, P_2) > 2M_1,$$

then

$$d_{Y_2}(P_1, R) \leq 2M_1$$

and

$$|d_{Y_2}(P_1, P_2) - d_{Y_2}(R, P_2)| \leq 2M_1,$$

and $Y_1 \prec_t Y_2$ in any hierarchy path $\rho(R, P_2)$.

Proof Since $Y_1 \prec_t Y_2$ in a hierarchy path $\rho(P_1, P_2)$ we have

$$d_{Y_1}(\partial Y_2, P_2) \leq M_1.$$

Since $2M_1 > M_2$, the subsurface Y_1 is a component domain of a hierarchy path from R to P_2. Arguing by contradiction suppose Y_2 is not a component domain of this latter path, or if it is, suppose $Y_2 \prec_t Y_1$. By the fourth property of hierarchies we would have

$$d_{Y_1}(R, \partial Y_2) \leq M_1.$$

The triangle inequality then gives

$$d_{Y_1}(R, P_2) \leq 2M_1,$$

a contradiction to the assumption.

Thus Y_2 is a component domain in a hierarchy path $\rho(R, P_2)$ and we have $Y_1 \prec_t Y_2$. Thus by the third property of hierarchies, $d_{Y_2}(\partial Y_1, R) \leq M_1$. Since $Y_1 \prec_t Y_2$ in $\rho(P_1, P_2)$ $d_{Y_2}(\partial Y_1, P_1) \leq M_1$ as well. The triangle inequality gives the first bound.

$$|d_{Y_2}(\partial Y_1, P_2) - d_{Y_2}(R, P_2)| \leq M_1$$

and

$$|d_{Y_2}(\partial Y_1, P_2) - d_{Y_2}(P_1, P_2)| \leq M_1.$$

The two inequalities together finish the proof. □

3 Relative stability of quasi-geodesics

We now specialize to the case when $S = S_{g,n}$ is a surface of genus g with n boundary components, and $\zeta(S) = 3$ (in other words, (g,n) is $(2,0)$, $(1,3)$, or $(0,6)$).

Definition 10 *Let $S = S_{g,n}$ where $\zeta(S) = 3$. Then we say an essential subsurface W of S is a* separated domain *if W is a four holed sphere or one-holed torus, and there is another four-holed sphere or one-holed torus W^c so that W and W^c may be embedded disjointly in S.*

Given a hierarchy path ρ, we denote by $X(\rho)$ the union of the image of ρ and the Farey-graph products $X_\gamma = C(W) \times C(W^c)$ where the separating curve $\gamma \in \mathcal{C}(S)$ is the common boundary of the separated domains W and W^c for which either W or W^c is a component domain of ρ.

Now Theorem 2 is a consequence of the following result, which guarantees the existence of a contracting projection map from $P(S)$ to $X(\rho)$.

We remind the reader of an important contraction property that always exists for projections to quasi-convex subsets of δ-hyperbolic metric spaces.

Definition 11 *A map Π has a (b,c)-contraction property if there exists $b, c > 0$ such that if $d(P,Q) < b d(\Pi(P), P)$ then $d(\Pi(P), \Pi(Q)) < c$.*

In particular, if X is a δ-hyperbolic metric space then there is a pair (b,c) so that if g is any geodesic lying in X then the nearest point projection $\pi_g : X \to g$ satisfies a (b,c)-contraction property. If a map Π satisfies a (b,c)-contraction property, for some b, c, we say it has the *contraction property*.

Theorem 12 *Fix a hierarchy path $\rho = \rho(P_1, P_2)$. There exists a projection map*

$$\Pi : P(S) \to X(\rho)$$

that is coarsely idempotent, coarsely Lipschitz, and has the contraction property.

Moreover the projection Π satisfies the following conditions: there is a constant K depending only on the topology of S so that for each $P \in P(S)$ we have

(1) *each non-separated component domain Z of ρ satisfies*

$$d_Z(\Pi(P), \pi_{gZ}(P)) < K,$$

(2) *each separated component domain W and its complement W^c satisfy*

$$d_W(\Pi(P), \pi_W(P)) < K, \;\; d_{W^c}(\Pi(P), \pi_{W^c}(P)) < K.$$

We refer to the final two conditions of the theorem as the *relative synchronization property* for the map Π. Notice that in condition (2) we do not require that the projection $\Pi(P)$ be close to any geodesic in $\mathcal{C}(W)$ or $\mathcal{C}(W^c)$.

Proof We break the proof into three parts. In part (i), we construct the projection. In part (ii), we verify that Π is relatively synchronized, coarsely idempotent and coarsely Lipschitz. Finally, in part (iii) we show the projection Π satisfies the contraction property.

Part (i): Constructing the Projection. Let $\rho\colon [0,n] \to P(S)$ be a hierarchy path with $\rho(0) = P_1$ and $\rho(n) = P_2$. Let

$$D(\rho) = S \cup \{Y : d_Y(P_1, P_2) \geq 2M_1\}$$

Given P in $P(S)$, let

$$D(P, \rho) = S \cup \{Y : \min(d_Y(P_1, P), d_Y(P, P_2)) \geq 2M_1\}.$$

First assume $D(P,\rho)$ contains a separated domain W, and either W or W^c is in $D(\rho)$. It follows immediately from Lemma 1 that there cannot be any other separated domain (other than W^c) with the same property. In this case set

$$\Pi(P) = \partial W \cup \pi_W(P) \cup \pi_{W^c}(P)$$

in the Farey graph product $X_{\partial W}$.

Now assume there is no such separating domain. Again from Lemma 1 two subsurfaces Y_1, Y_2 in $D(P,\rho) \cap D(\rho)$ cannot transversely intersect. Then we let $B(P,\rho)$ denote the union of all the isotopy classes of boundary components of the subsurfaces in $D(P,\rho) \cap D(\rho)$ that are non-peripheral in S. We note that $B(P,\rho)$ may be empty. If it is nonempty it is a simplex in $\mathcal{C}(S)$. If $B^*(P,\rho)$ denotes the realization of these isotopy classes as a disjoint union of simple closed curves on S then consider the complement $S \setminus N(B^*(P,\rho)))$ of a union of annular neighborhoods of these curves. If this complement contains a subsurface Y that is not a 3-holed sphere, then Y is unique, since it cannot be a separated domain by assumption, and furthermore Y lies in $D(P,\rho) \cap D(\rho)$. In this case Y is minimal in $D(P,\rho) \cap D(\rho)$ with respect to the order induced by inclusion (we remind the reader that Y may be the full surface S).

Recall k is the Lipschitz constant for the projection π_{g_W}. Denote by $\mathcal{W}(P,\rho)$ the (possibly empty) set of all $W \subset Y$ such that

- $d_W(P_1, P_2) > 10kM_1$.
- $I_W \cap I_Y \neq \emptyset$

Note in particular that the first condition implies $W \in D(\rho)$. Since Y is minimal in $D(P, \rho)$, we have $W \notin D(P, \rho)$ so $d_W(P, P_i) \leq 2M_1$ for either P_1 or P_2. Then we claim that the following order relationship holds. Either

(1) $d_W(P_1, \pi_{g_W}(P)) \leq 5kM_1$ and we say $P < W$ or

(2) $d_W(\pi_{g_W}(P), P_2) \leq 5kM_1$ and we say $P > W$.

(The definition of $\mathcal{W}(P, \rho)$ and the triangle inequality says that both inequalities cannot hold). To prove the claim assume without loss of generality that $d_W(P, P_1) \leq 2M_1$. Let v_0 be the initial vertex of the geodesic g_W. By property 4 of hierarchy paths, we have $d_W(P_1, v_0) \leq M_1$, and so by the triangle inequality,

$$d_W(P, v_0) \leq 3M_1.$$

Since $\pi_{g_W}(P)$ is the closest vertex in g_W we then have

$$d_W(P, \pi_{g_W}(P)) \leq 3M_1$$

and so again by the triangle inequality

$$d_W(P_1, \pi_{g_W}(P)) \leq 5M_1 \leq 5kM_1$$

proving the claim.

Let $\mathcal{W}_-$ denote the set of $W \in \mathcal{W}(P, \rho)$ with $P > W$ and let $\mathcal{W}_+$ denote the set of $W \in \mathcal{W}(P, \rho)$ with $P < W$.

Define the subsets of the parameter values

$$U_- = \bigcup_{W \in \mathcal{W}_-} I_W \quad \text{and} \quad U_+ = \bigcup_{W \in \mathcal{W}_+} I_W.$$

Then we let i_1' be the maximum parameter value in U_- and i_2' be the minimum parameter value in U_+. (If $U_- = \emptyset$ take i_1' to be the initial point of I_Y and similarly if $U_+ = \emptyset$ take i_2' the maximal value of I_Y). Let

$$P_1' = \rho(i_1') \quad \text{and} \quad P_2' = \rho(i_2').$$

There is a connected set J of vertices on g_Y that are vertices of pants decompositions between $P_1' = \rho(i_1')$ and $P_2' = \rho(i_2')$. By the hyperbolicity of $\mathcal{C}(Y)$ there are a bounded number of vertices in J that are closest to the projection of P into $\mathcal{C}(Y)$. Pick one such closest v. Let $\rho(j)$ be a pants decomposition that contains this v and define

$$\Pi(P) = \rho(j).$$

This completes the construction.

Part (ii): The map Π is relatively synchronized, coarsely idempotent, and coarsely Lipschitz.

We first note the following Lemma. Let Y be the surface determined by the construction of Π. If Y is a proper subsurface, then $\Pi(P)$contains ∂Y.

Lemma 2 *Suppose Y is a proper subsurface. Suppose Z transversely intersects Y, and $d_Z(P_1, P_2) \geq 4M_1$. Then $d_Z(P, \Pi(P)) \leq 4M_1$.*

Proof If the conclusion of the lemma is false, then Z is a component domain of a hierarchy $\rho(P, \Pi(P))$. Since $\Pi(P)$ contains ∂Y, by the fourth property of hierarchies, $d_Y(\partial Z, P) \leq M_1$. On the other hand, by the triangle inequality, for either $i = 1, 2$ we have $d_Z(P_i, \Pi(P)) \geq 2M_1$. Without loss of generality assume this holds for $i = 1$. Then again by the fourth property of hierarchies, since $\Pi(P)$ contains ∂Y, $d_Y(P_1, \partial Z) \leq M_1$ and so by the triangle inequality we have

$$d_Y(P_1, P) \leq 2M_1,$$

a contradiction to the fact that $Y \in D(P, \rho)$. □

We now give the proof of relative synchronization. First assume that Z is not separating. The proof breaks into cases.

Case I. *The subsurface Z transversely intersects Y.* If $d_Z(P_1, P_2) \leq 4M_1$, then by property 4 of hierarchies and the fact g_Z is a geodesic,

$$d_Z(\Pi(P), \pi_{g_Z}(P)) \leq 5M_1$$

and we can take $K = 5M_1$.

If $d_Z(P_1, P_2) \geq 4M_1$, then by Lemma 2 we have $d_Z(P, \Pi(P)) \leq 4M_1$. But then the fact that projections to geodesics are k-Lipschitz says that

$$d_Z(\pi_{g_Z}(\Pi(P)), \pi_{g_Z}(P)) \leq 4kM_1$$

By Property 4 of hierarchies, $d_Z(\pi_{g_Z}(\Pi(P)), \Pi(P)) \leq M_1$. Therefore by the triangle inequality, we have the result for $K = 4kM_1 + M_1$.

Case II. $Z = Y$ If we can show that some closest point projection $v' = \pi_{g_Y}(P)$ satisfies

$$d_Y(v', J) \leq 3,$$

then by the hyperbolicity of $\mathcal{C}(Y)$, certainly v' would be within bounded distance of $\Pi(P)$, the nearest point projection in J of P, and we would be done. Assume this is false, and assume without loss of generality that all closest v' lie before J along g_Y at distance greater than 3.

Let $W \in W_-$ be the domain such that the right endpoint of I_W coincides with the left endpoint of J. Let h be the geodesic in $\mathcal{C}(Y)$ joining $\pi_Y(P)$ to v'. Then we first assert that every vertex of h intersects ∂W. If this were not true then some vertex u of h would be within distance 1 of ∂W and therefore within distance 2 of J.

If $d_S(u, v') \leq 1$ then $d_S(J, v') \leq 3$, a contradiction to the assumption. If $d_S(u, v') \geq 2$, then $d_S(u, J) \leq d_S(u, v')$, and we have contradicted that there are no closest points in J. This proves the assertion that every vertex of h must intersect ∂W. But then by the first property of hierarchy paths,

$$d_W(P, v') \leq M_2.$$

By the fourth property of hierarchies,

$$d_W(P_1, v') \leq M_1.$$

By the triangle inequality, $d_W(P, P_1) \leq M_1 + M_2 \leq 2M_1$. Then the Lipschitz property of projections to geodesics implies

$$d_W(\pi_{g_W}(P), \pi_{g_W}(P_1)) \leq 2kM_1$$

and so again by the fourth property of hierarchies and the triangle inequality,

$$d_W(P_1, \pi_{g_W}(P)) \leq 2kM_1 + M_1 < 5kM_1$$

so $P < W$. This contradiction finishes the argument.

Case III. *The subsurface Y is a proper subsurface of Z.* Since $\Pi(P)$ contains ∂Y it is enough to show that

$$d_Z(\partial Y, \pi_{g_Z}(P)) < 3.$$

Assuming otherwise, since $\pi_{g_Z}(P)$ is the closest point projection, every vertex on the geodesic h in $\mathcal{C}(Z)$ joining $\pi_Z(P)$ to $\pi_{g_Z}(P)$ intersects Y. Consequently, by the first property of hierarchies, we have $d_Y(P, \pi_{g_Z}(P)) \leq M_2$. Since $\pi_{g_Z}(P)$ intersects Y, for either $i = 1, 2$ $d_Y(P_i, \pi_{g_Z}(P)) \leq M_1$, so for that value of i,

$$d_Y(P, P_i) \leq M_1 + M_2 \leq 2M_1$$

a contradiction to $Y \in D(P, \rho)$.

Case IV. *The subsurface Z is a proper subsurface of Y.* In this case Z belongs to $\mathcal{W}$. Without loss of generality assume $P > Z$ so that

$$d_Z(P_2, \pi_{g_Z}(P)) \leq 5kM_1.$$

Since

$$d_Z(\Pi(P), P_2) \leq M_1$$

by the fourth property of hierarchies, we are done by the triangle inequality. We have proved 1.

We now establish 2. If the separating W lies in $D(P, \rho)$ and either W or W^c lies in $D(\rho)$, then $\Pi(P)$ lies in X_γ where $W \subset S \setminus \gamma$, and in this case $d_W(P, \Pi(P)) = 0$. The cases that $W \notin D(P, \rho)$ or both $W, W^c \notin D(\rho)$ are part of 1.

It follows from relative synchronization that the map is coarsely idempotent; there is a constant K such that if $P \in \rho[0, n]$ then $d(\Pi(P), P) \leq K$.

To see that the map is coarsely Lipschitz, we note that the image depends only on the projections of P to subsurfaces of S. Since projections are Lipschitz maps, there is a constant C such that if $d_W(P, P') = 1$, then $d_W(\Pi(P), \Pi(P')) \leq C$. Now the Lipschitz property follows from (2.1).

□

Part (iii): The projection Π has the contraction property.

We begin this section by establishing further properties of the projection map Π. Let $\rho' : [0, N] \to P(S)$ denote the hierarchy joining P and $\Pi(P)$. The next lemma says that Π is almost a nearest point projection on ρ.

Lemma 3 *Suppose Z is a component domain of ρ' with corresponding geodesic g'_Z and parameter interval I'_Z. Let j'_Z be the last parameter value and k'_Z any parameter value.*

- *If Z is also a component domain of ρ with geodesic g_Z, then*

  $$d_Z(\pi_{g_Z}(P_{k'_Z}), \pi_{g_Z}(P_{j'_Z})) \leq 2kM_1 + K + M_1 + 2k\delta.$$

  where $\mathcal{C}(Z)$ is δ hyperbolic.

- *The closest point projection $\pi_{g'_Z}$ satisfies*

  $$d_Z(\pi_{g'_Z}(P), P_{j'_Z}) \leq (4 + 2k)\delta + 2M_1(2k + 1) + K + M_2$$

  for all $P \in \rho$.

Proof Let i'_Z be the initial parameter value of I'_Z. By the fourth property of hierarchies, we have

$$d_Z(P, P_{i'_Z}) \leq M_1 \quad \text{and} \quad d_Z(\Pi(P), P_{j'_Z}) \leq M_1. \tag{3.2}$$

Since the projections to a geodesic is k-Lipschitz, we get

$$d_Z(\pi_{g_Z}(P), \pi_{g_Z}(P_{i'_Z})) \leq kM_1 \quad \text{and} \quad d_Z(\pi_{g_Z}(\Pi(P)), \pi_{g_Z}(P_{j'_Z})) \leq kM_1. \tag{3.3}$$

Again applying the fourth property of hierarchies, we have

$$d_Z(\Pi(P), \pi_{g_Z}(\Pi(P))) \leq M_1. \tag{3.4}$$

Thus by relative synchronization we have

$$d_Z(\pi_{g_Z}(P), \pi_{g_Z}(\Pi(P))) \leq K + M_1. \tag{3.5}$$

The triangle inequality applied to inequalities (3.3) and (3.5) gives

$$d_Z(\pi_{g_Z}(P_{i'_Z}), \pi_{g_Z}(P_{j'_Z})) \leq 2kM_1 + K + M_1.$$

The first conclusion which is stated for any parameter value now follows from the hyperbolicity of $\mathcal{C}(Z)$ and the fact that projections are Lipschitz.

We prove the second statement. Notice first that if Z is not component domain of ρ, then by the first and fourth property of hierarchies, and the triangle inequality.

$$d_Z(P, P_{j'_Z}) \leq M_2 + M_1$$

and the projection of P to g'_Z is even closer.

Now suppose Z is a component domain with geodesic g_Z. Let v be any vertex of g_Z and let $v' = \pi_{g'_Z}(v)$ its closest point projection on g'_Z. Let $v'' = \pi_{g_Z}(v')$ the closest point projection of v' on g_Z. The first conclusion of the Lemma, the fourth property of hierarchies, and the hyperbolicity of $\mathcal{C}(Z)$ imply that

$$d_Z(P_{j'_Z}, v'') \leq 2\delta + 2kM_1 + K + M_1 + 2k\delta + M_1.$$

Since v'' is the closest point to v' on g_Z, the δ hyperbolicity of $\mathcal{C}(Z)$ implies that any geodesic h joining v' to v must pass within 2δ of v'' and therefore by the above bound, within $(4 + 2k)\delta + 2M_1(k + 1) + K$ of $\pi_{g_Z}(P_{j'_Z})$. This gives the bound for $d_Z(v, v')$. Since any $P \in \rho$ is distance at most M_1 from g_Z and the projections to geodesics are k Lipschitz, the result now follows from the triangle inequality.

□

Lemma 4 *There exists a constant K'' such that for any pants decomposition $Q' = \rho'(i)$ in the hierarchy $\rho(P, \Pi(P))$,*

$$d(\Pi(Q'), \Pi(P)) \leq K'.$$

Proof We bound $d(\Pi(P), \Pi(Q'))$ using the distance formula (2.1). Since Π is Lipschitz we need only bound $d_Z(\Pi(P), \Pi(Q'))$ for those Z that are component domains of both ρ and the sub-hierarchy of ρ' joining P and Q'.

Now Lemma 3 says that the geodesic g'_Z has bounded diameter projection to g_Z. This together with the fourth property of hierarchies and the fact that projection to g_Z is Lipschitz bounds the projection of any two points in the sub-hierarchy.

□

Now let Π' denote the projection to the hierarchy ρ'.

Lemma 5 *There is K''' and C, such that if Z is a subsurface and Q is a pants decomposition such that*

$$d_Z(\Pi'(Q), \Pi(P)) \geq C,$$

then

(1) $d_Z(Q, \Pi(P)) \geq 2M_1$.

(2) $d_Z(Q, \Pi(Q) \geq 2M_1 + K$.

(3) $d_Z(\Pi(P), \Pi(Q)) \leq K'''$.

(4) *For any W, if $d_W(Q, \Pi'(Q)) \geq C + 2M_1$ then $d_W(Q, \Pi(P)) \geq 2M_1$.*

Proof Let $Q' = \Pi'(Q)$. In order to bound the distance in $\mathcal{C}(Z)$ between the projections of $\Pi(P)$ and $\Pi(Q)$, we can assume that Z is a component domain of ρ with geodesic g_Z. We consider the geodesic g in $\mathcal{C}(Z)$ joining $\pi_Z(\Pi(P))$ and $\pi_Z(\Pi(Q))$. By the fourth property of hierarchies and the hyperbolicity of $\mathcal{C}(Z)$, there is a constant δ' such that the geodesic g is Hausdorff distance δ' from g_Z. By relative synchronization and Lemma 3, there is a δ'' such that the closest point projection of $\pi_Z(Q')$ on g is within δ'' of $\pi_Z(\Pi(P))$ and the closest point projection of $\pi_Z(Q)$ on g is within δ'' of $\pi_Z(\Pi(Q))$.

We consider the quadrilateral in $\mathcal{C}(Z)$ with vertices

$$\{\pi_Z(Q), \pi_Z(Q'), \pi_Z(\Pi(P)), \pi_Z(\Pi(Q))\}.$$

The hyperbolicity of $\mathcal{C}(Z)$ implies that there are constants C and δ''' depending on δ' and δ'' so that for $d_Z(Q', \Pi(P)) > C$ we have

$$d_Z(Q, \Pi(P)) \geq 2M_1 \quad \text{and} \quad d_Z(Q, \Pi(Q)) \geq 2M_1 + K,$$

and the geodesic h joining $\pi_Z(Q)$ and $\pi_Z(\Pi(Q))$ passes within δ''' of $\Pi(P)$. We will choose C so that

$$C > (4 + 2k)\delta + 2M_1(2k + 1) + K + M_2, \tag{3.6}$$

the constant on the right side coming from Lemma 3. Taken together with the fact that closest point projection of $\pi_Z(Q)$ on g is within δ'' of $\pi_Z(\Pi(Q))$, we obtain the bound

$$d_Z(\Pi(P), \Pi(Q)) < 2\delta''' + \delta''$$

by an application of the triangle inequality. Setting $K''' = 2\delta''' + \delta''$, we have the first three statements of the Lemma.

To prove the last statement, notice again by the first part that for $d_W(Q', \Pi(P)) \geq C$, we have

$$d_W(Q, \Pi(P)) \geq 2M_1.$$

If $d_W(Q', \Pi(P)) \leq C$, then the assumption $d_W(Q, Q') \geq C + 2M_1$ gives

$$d_W(Q, \Pi(P) \geq 2M_1$$

by the triangle inequality. □

Proof of Theorem 12 To conclude the proof of Theorem 12 we seek a pair (b, c) so that the map Π satisfies an (b, c)-contraction property. Choose b so that

$$b < \frac{1}{2k'}$$

where k' is the Lipschitz constant for the map Π'. Then since $\Pi'(P) = P$ we have

$$\begin{aligned} d(P, \Pi'(Q)) &= d(\Pi'(P), \Pi'(Q)) \\ &\leq k' d(P, Q) \\ &\leq bk' d(P, \Pi(P)) \\ &< \frac{d(P, \Pi(P))}{2}. \end{aligned}$$

This implies

$$d(\Pi'(Q), \Pi(P)) > \frac{d(P, \Pi(P))}{2}.$$

Set $Q' = \pi'(Q)$. There is a constant a_0 such that if $d(P, \Pi(P)) \geq a_0$, the above inequality and the distance formula (2.1) guarantees the existence of a domain Z for the hierarchy path ρ' such that

$$d_Z(Q', \Pi(P)) \geq C + 2M_1 + K,$$

where C is the constant given by Lemma 5. Assume then that

$$d(P, \Pi(P)) \geq a_0.$$

Let $g'Z$ the geodesic in $\mathcal{C}(Z)$ in ρ', recall $g'Z$ has terminal parameter value $j_{Z'}$. Then the fourth property of hierarchies implies

$$d_Z(Q', P_{j'_Z}) \geq C + M_1 + K \tag{3.7}$$

By the distance formula (2.1), bounding $d(\Pi(P), \Pi(Q))$ is equivalent to bounding $d_W(\Pi(P), \Pi(Q))$ for each component domain W of ρ and by Lemma 4 this is equivalent to bounding $d_W(\Pi(Q), \Pi(Q'))$. We first note that if

$$d_W(Q, Q') \leq C + 2M_1 + K$$

then $d_W(\Pi(Q), \Pi(Q')) \leq k'(C + 2M_1 + K)$, since the map Π is k'-Lipschitz.

Thus we can restrict to domains W with $d_W(Q, Q') \geq C + 2M_1 + K$.

Claim 13 *If $d_W(\Pi(P), \Pi(Q)) \geq 2M_1$, i.e. $W \in D(\rho)$, then $Z \subseteq W$.*

By way of contradiction first assume W transversely intersects Z. By the last conclusion of Lemma 5 we have

$$d_W(Q, \Pi(P)) \geq 2M_1$$

and

$$d_Z(Q, \Pi(P)) \geq 2M_1.$$

Since W and Z transversely intersect, they are time ordered in any hierarchy $\rho(Q, \Pi(P))$. If $Z \prec_t W$, then since $d_Z(Q', \Pi(P)) \geq 2M_1$, an application of Lemma 1 with $Q' = R$ from that lemma says that $d_W(Q, Q') \leq 2M_1$, a contradiction.

Thus $W \prec_t Z$ in $\rho(Q, \Pi(P))$. We apply Lemma 1 again using the hierarchy $\rho(Q, \Pi(P))$ this time with $R = \Pi(Q)$. Since $d_Z(Q, \Pi(Q)) \geq 2M_1$ we conclude that $d_W(\Pi(P), \Pi(Q)) \leq 2M_1$, a contradiction. We have ruled out W transversely intersecting Z.

Next suppose $W \subsetneq Z$. By the K relative synchronization of the projection Π', we have,

$$d_Z(Q', \pi_{g'_Z}(\partial W)) \leq K$$

so by (3.7) and the triangle inequality,

$$d_Z(\pi_{g'_Z}(\partial W), P_{j'_Z}) \geq C + M_1.$$

On the other hand, since W is assumed to be a component domain of ρ, by the second conclusion of Lemma 3

$$d_Z(\pi_{g'_Z}(\partial W), P_{j'_Z}) \leq (4+2k)\delta + 2M_1(2k+1) + M_2 + K < C$$

and again we have a contradiction. This proves the claim.

By the claim then we need only bound $d_W(\Pi(Q), \Pi(Q'))$ for $Z \subseteq W$. By third conclusion of Lemma 5 we have the bound for $W = Z$. The remaining possibility is W contains Z. By relative synchronization,

$$d_W(\Pi(P), \pi_{g_W}(\partial Z)) \leq K, \;\; d_W(\Pi(Q), \pi_{g_W}(\partial Z)) \leq K.$$

The desired bound $d_W(\Pi(P), \Pi(Q))) \leq 2K$ is given by the triangle inequality.

Finally we consider the possibility that $d(P, \Pi(P)) \leq a_0$ so that $d(P,Q) \leq a_0 b$ by assumption. Then since Π is $\hat{k}$ Lipschitz for some $\hat{k}$ we get a bound

$$d(\Pi(P), \Pi(Q)) \leq \hat{k} a_0 b$$

in this case. Taking $c = \max(2K, \hat{a}_0 b)$ we get the desired bound.

□

Proof of Theorem 2 This follows from the usual Mostow type argument. A proof in this context is given by Lemma 7.1 of [MM1]: by Theorem 12 a quasi-geodesic cannot stray far from the set $X(\rho)$ due to inefficiency outside of a bounded neighborhood of the projection image. □

The notion of *strong relative hyperbolicity* with respect to a collection of subsets, introduced for groups by Farb [Fa] (see also [BF] for a metric space notion in a similar context to this paper) finds its currently accepted form in [DS], and [Dru].

Definition 14 *A metric space is said to be* strongly relatively hyperbolic *with respect to a collection of subsets $\mathcal{H}$ if*

(1) *Given K there exists M such that the intersection of K neighborhoods of any two subsets from $\mathcal{H}$ is M bounded.*

(2) *Given L, C, there is M such that for any pair of points x, y and subset A from the collection $\mathcal{H}$, if $d(x,y) \geq 3\max(d(x,A), d(y,A))$, then any (L,C) quasi-geodesic between x, y crosses the M neighborhood of A.*

(3) *For every k, there exists M such that every thick k-gon belongs to one of the sets in $\mathcal{H}$.*

By work of Druţu [Dru], the last condition can be replaced by

3^* *For positive constants L and C there are constants M and M', such that for any (L,C)-quasi-geodesic triangle in X, there exists a set A in the collection $\mathcal{H}$ whose M neighborhood intersects the three sides of the triangle, such that the pairs of entrance points of the sides in this neighborhood starting from the same vertex are distance at most M' apart.*

The proof of Theorem 1 hinges on the following Proposition, verifying a condition reminiscent of Farb's bounded coset penetration property of [Fa] and [BF].

Proposition 1 *Given K, C there are K_1, K_2 with the property that if ϕ is any (K,C)-quasigeodesic joining pants decompositions, P_1 and P_2, and $\rho = \rho(P_1, P_2)$ is a hierarchy joining P_1, P_2, then for any for any Farey graph product $X_\gamma \subset X(\rho)$, the path ϕ and the hierarchy ρ enter and exit a K_1 neighborhood of X_γ within distance K_2 of each other.*

Proof By Theorem 2, given (K,C), there exists K' such that any (K,C)-quasi-geodesic $\phi(n)$ stays within K' of $X(\rho)$ where $\rho = \rho(P_1, P_2)$. By relative synchronization and the distance formula (2.1) there is a constant K_1 depending on K' such that for all n,

$$d(\phi(n), \Pi(\phi(n))) \leq K_1.$$

Now given a Farey graph product $X_\gamma \subset X(\rho)$ with complementary domains $X \setminus \gamma = W \cup W^c$, let n be the maximal parameter value such that for all $j = 1, \ldots, n$, $\phi(j)$ does not lie in $\mathcal{N}_{K_1}(X_\gamma)$, the K_1 neighborhood of the product X_γ.

Let I_W, I_{W^c} the parameter intervals for W, W^c. (at least one of which is nonempty). Now let $j \leq n$. If

$$\max(|I_W|, |I_{W^c}|) \geq k,$$

we claim that either $\Pi(\phi(j)) \in X_{\gamma'}$ where some complementary component of γ' is time ordered before W or W^c, or $\Pi(\phi(j)) = \rho(j')$ where j' satisfies

$$j' < I_W \cup I_{W^c}.$$

Suppose on the contrary, there exists $j_1, j_2 \leq n$ such that

$$\Pi(\phi(j_1)) \in X_{\gamma_1} \cup \rho(j')$$

for some complementary component of γ_1 time ordered before W or W^c and $j' < I_W \cup I_{W^c}$ and

$$\Pi(\phi(j_2)) \in X_{\gamma_2} \cup \rho(j'')$$

for some complementary component of γ_2 time ordered after W or W^c and $j'' > I_W \cup I_{W^c}$.

Since X_γ separates $X(\rho)$ and $\Pi(\phi(j_1))$ and $\Pi(\phi(j_2))$ lie in distinct components of $X(\rho) \setminus X_\gamma$, any path joining $\Pi(\phi(j_1))$ and $\Pi(\phi(j_2))$ must enter X_γ. Since the map Π is k-Lipschitz and $\max(|I_W|, |I_{W^c}|) \geq k$, there must be some $j_1 < j_0 < j_2$ such that $\Pi(\phi(j_0)) \in X_\gamma$, contrary to assumption, proving the claim.

Then since $d(\Pi(\phi(j)), \phi(j)) \leq K_1$ it follows from (2.1) and the fourth property of hierarchies, that for all $j \leq n$,

$$d_W(\phi(j), P_1) \leq K_1 + M_1 \quad \text{and} \quad d_{W^c}(\phi(j), P_1) \leq K_1 + M_1.$$

This implies that the first time ϕ enters a K_1 neighborhood of X_γ (at time $n+1$) there is a bound on $d_W(P_1, \phi(n+1))$ and $d_{W^c}(P_1, \phi(n+1))$ and this in turn, by the fourth property of hierarchies implies that ϕ enters this neighborhood a bounded distance K_2 from where ρ does. The same is true for the minimal parameter value m where $\phi(j) \notin \mathcal{N}_{K_1}(X_\gamma)$ for all $j \geq m$.

For those X_γ such that $\max(|I_W|, |I_{W^c}|) \leq k$ we simply note that the bound $d(\phi(n), \Pi(\phi(n))) \leq K_1$ bounds

$$\max\left(d_W(\phi(n), P_i), d_{W^c}(\phi(n), P_i)\right).$$

In particular, $\phi(n)$ enters any neighborhood of X_γ a uniformly bounded distance K_2 away from where $\rho(n)$ does. This completes the proof. □

Proof of Theorem 1 We show property (1) is satisfied. Let $X_\gamma, X_{\gamma'}$ a pair of Farey graph products. Let W and W^c the separating domains with boundary γ. Since γ' intersects both W and W^c, and every curve in the Farey graph product $X_{\gamma'}$ is disjoint from γ', the projection of the entire Farey graph product X_γ to W and W^c lies at bounded distance from the projection of γ'. Since the projection map is Lipschitz, the same is true of a K neighborhood of $X_{\gamma'}$. This together with the distance formula (2.1) gives the first condition.

To verify (2), we can assume

$$d(x, y) \geq 6K'(M_1 + M_2) + 3K'C.$$

Again let W and W^c be the components of the complement of γ. Let $a \in X_\gamma$ the closest point to x and $b \in X_\gamma$ the closest point to y. By the fourth property of hierarchies,

$$d_W(x,a) \leq M_1, \quad d_{W^c}(x,a) \leq M_1$$

with the same inequalities with y and b. Then we have

$$\begin{aligned} d_W(x,y) + d_{W^c}(x,y) &\geq d_W(a,b) + d_{W^c}(a,b) - 2M_1 \\ &\geq \frac{d(a,b)}{K'} - C - 2M_1 \\ &\geq \frac{d(x,y)}{3K'} - C - 2M_1 \\ &\geq 2M_2 \end{aligned}$$

Then by the first property of hierarchies, any hierarchy joining x and y passes through X_γ, and therefore by Proposition 1 any (L,C)-quasi-geodesic passes through a bounded neighborhood of X_γ. If

To verify (3^*), without loss of generality assume $d_W(x,y) \geq M_2$ for some separated domain W. Then any hierarchy joining x and y passes through the corresponding Farey graph product X_γ at a point a that satisfies $d_W(x,a) \leq M_1$ and $d_{W^c}(x,a) \leq M_1$. If $d_W(x,z) \geq M_2$ or $d_{W^c}(x,z) \geq M_2$ the same estimates hold for the entry point b for any hierarchy joining x and z. The triangle inequality then bounds the distance between a and b in these domains. If a hierarchy joining x and z does not enter the Farey graph product, one has the bound M_2 on $d_W(x,z)$ and $d_{W^c}(x,z)$. Thus in either case we have a bound on the distance between a and b in a fixed neighborhood of X_γ. Again by Proposition 1 this is true for any (L,C)-quasi-geodesic. □

Proof of Theorem 3 Suppose $\phi : \mathbb{R}^2 \to P(S)$ is a (K_0, C_0)-quasi-isometric embedding. For each $R > 0$ join $\phi(-R,0)$ to $\phi(R,0)$ by a hierarchy path ρ_R. Now consider for $|R_1| \leq R$, the $\sqrt{2}$ quasi-geodesic σ_{R,R_1} in $\mathbb{R}^2$ consisting of segments joining $(-R,0)$ to $(-R,R_1)$, $(-R,R_1)$ to (R,R_1) and (R,R_1) to $(R,0)$.

Its image $\phi(\sigma_{R,R_1})$ is a $K_0\sqrt{2}+C_0$ quasi-geodesic joining $\phi(-R,0)$ and $\phi(R,0)$. Since ϕ is a (K_0,C_0)-quasi-isometry, except for an initial and final segment on each of length R'

$$d(\phi(\sigma_{R,R_1}), \rho_R) > \frac{R'}{K_0} - C_0.$$

Since $\phi(\sigma_{R,R_1})$ must remain a bounded distance from $X(\rho_R)$ it follows that for $\frac{R'}{K_0} - C_0$ sufficiently large, by Theorem 1 these points on $\phi(\sigma_{R,R_1})$ must lie in a bounded neighborhood of the product of Farey graphs $\{X_\gamma \mid \gamma \in \rho_R\}$.

For R' chosen sufficiently large, but fixed, the fact that bounded neighborhoods of Farey graph products have bounded intersection guarantees that this subset of $\phi(\sigma_{R,R_1})$ must lie within a bounded neighborhood of a single X_γ, and in addition, $\phi(\sigma_{R,R_1})$ must enter and exit this neighborhood a bounded distance from where ρ_R enters and exits the neighborhood. The above statement is true for arbitrary R, and therefore by enlarging R while keeping R_1 fixed, we conclude there is a single X_γ such that the image of the entire horizontal line $y = R_1$ lies in a fixed neighborhood of X_γ. Since this is true for arbitrary R_1 larger than a fixed size, the entire quasi-flat must lie within a bounded distance of a single X_γ. □

Proof of Corollary 4 We claim there is no quasi-isometric embedding

$$\varphi\colon \mathbb{R}^3 \to P(S).$$

To see this, note that by Theorem 3 φ maps the x-y plane to a bounded neighborhood of a single Farey graph product X_γ and the y-z plane to a bounded neighborhood of a product $X_{\gamma'}$. Since these planes meet along a line, we have that X_γ and $X_{\gamma'}$ have uniform neighborhoods with infinite diameter intersection, which implies that $\gamma = \gamma'$. Composing with the nearest point projection to X_γ then, we have a quasi-isometric embedding of $\mathbb{R}^3$ into a product of Farey-graphs, which is impossible by [KL].

□

4 The boundary of the Weil-Petersson metric

In low complexity cases, when the pants graph is Gromov hyperbolic, the collection of asymptote classes of geodesic rays is basepoint invariant, and corresponds to the usual *Gromov boundary* of the Gromov hyperbolic space.

For a general CAT(0) space the asymptote class of an infinite geodesic ray remains a basepoint invariant notion (see [BH]). We find that the relative stability of quasi-geodesics in $P(S)$ when $\zeta(S) = 3$ provides for sufficient control over geodesic rays in the Weil-Petersson metric to give a description of the CAT(0) boundary in this setting as well.

To this end, we briefly recall some standard properties of the the Weil-Petersson metric and its completion. For more details, we direct the reader to [Wol2] and [Br1].

The Weil-Petersson completion. It is due to Wolpert and Chu that the Weil-Petersson metric is not complete. Masur examined the structure of the completion $\overline{\text{Teich}(S)}$ and

found a natural correspondence between the completion and the *augmented Teichmüller space*, consisting of marked Riemann surfaces with nodes corresponding to a pairwise disjoint collection of simple closed curves on S that have been pinched.

The augmented Teichmüller space has a stratified structure organized by simplices in the curve complex. This structure is most easily described via the notion of *extended Fenchel-Nielsen coordinates* as follows: given a maximal simplex σ in the curve complex $\mathcal{C}(S)$, with vertex set $\sigma^\circ = \{\alpha_1, \ldots, \alpha_{\zeta(S)}\}$ the usual associated length-twist Fenchel-Nielsen coordinates for a surface $X \in \text{Teich}(S)$ are given by the product

$$\Pi_{\alpha \in \sigma^\circ}(\ell_\alpha(X), \theta_\alpha(X)) \in \mathbb{R}_+^{\zeta(S)} \times \mathbb{R}^{\zeta(S)}$$

indicating that X is assembled from hyperbolic three-holed spheres with geodesic boundary lengths $\ell_\alpha(X)$ and twist parameters $\theta_\alpha(X)$, $\alpha \in \sigma^\circ$ (see [IT]). Then the extended Fenchel-Nielsen coordinates corresponding to σ parametrize the subset of $\overline{\text{Teich}(S)}$ with 'nodes along σ' by allowing the parameters $\ell_\alpha(X) = 0$, and imposing the equivalence relation

$$(0, \theta) \sim (0, \theta')$$

for coordinates $(\ell_\alpha, \theta_\alpha)$ where ℓ_α vanishes.

For any sub-simplex $\eta \subset \sigma$, then, the *η-stratum* $\text{Teich}_\eta(S)$ refers to the locus

$$\{\ell_\alpha(X) = 0 \iff \alpha \in \eta^\circ\},$$

in other words, the subset of $\overline{\text{Teich}(S)}$ where precisely the curves corresponding to the vertices of η have vanishing length functions. The η-stratum has the natural structure of a (possibly empty) product of Teichmüller spaces of the complementary subsurfaces $Y \subset S$ of non-zero Teichmüller dimension in the complement $S \setminus \{\alpha_1, \ldots, \alpha_k\}$ of the simple closed curves $\{\alpha_i\}$ corresponding to the vertices of η.

Asymptote classes and hierarchy paths. We first remark that there is a natural invariant of the asymptote class of a half-infinite hierarchy path $\rho(n)$, $n \in \mathbb{N}$, in the pants graph, which we will call a *boundary lamination*. Indeed, the collection of subsurfaces $W \subset S$ for which

$$\text{diam}_W(\rho([0, n])) \to \infty$$

as $n \to \infty$ form a pairwise disjoint collection of subsurfaces of S. Since each is a component domain for the hierarchy path $\rho(n)$, each carries a geodesic $g_W \subset \mathcal{C}(W)$ of infinite length. This geodesic is asymptotic to a geodesic lamination λ_W (filling W) in the Gromov boundary $\partial\mathcal{C}(W)$ (see [Kla]), so that for all n sufficiently large, each $\rho(n)$ contains a curve in g_W.

By the distance formula 2.1, if $\rho'(n)$ is another hierarchy path for which

$$d(\rho(n), \rho'(n)) < D$$

for all n, then $d_W(\rho(n), \rho'(n)) < D'$ for some D', from which it follows that $\rho'(n)$ contains curves in $\mathcal{C}(W)$ asymptotic to λ_W. Hence the union of these λ_W forms a geodesic lamination on S which is an invariant of the asymptote class of $\rho(n)$.

We note that the case when $\zeta(S) \leq 2$, the asymptote class is uniquely determined by this lamination, since in these low complexity cases the boundary lamination associated to a hierarchy path is connected, and any two hierarchy paths with the same boundary lamination lie a bounded distance apart: this can be seen directly from property 4 of the definition of hierarchy paths. The distance formula 2.1, guarantees that for any proper subsurface $Y \subsetneq S$, the projections $\pi_Y(\rho(n))$ begin at $\pi_Y(\rho(0))$ and lie at a bounded distance from the geodesic joining $\pi_Y(\rho(0))$ to $\pi_Y(\lambda)$ (where this geodesic is infinite if λ is a lamination in Y).

To relate this discussion to Weil-Petersson geodesics, we begin by associating to each a hierarchy path ρ (via Theorem 2) and then associating to that the corresponding boundary lamination for the asymptote class of ρ in $P(S)$.

Let $\{X(t)\}_{t=0}^{\infty}$ be a geodesic in the Weil-Petersson metric. Then by Theorem 1 of [Br1] the geodesic $X(t)$ describes a quasi-geodesic $\{P_n\}_{n=0}^{\infty}$ in $P(S)$ by taking its image under the quasi-isometry

$$Q\colon \mathrm{Teich}(S) \to P(S).$$

By Theorem 2, there is a hierarchy path $\rho(n)$, so that the quasi geodesic $\{P_n\}$ stays a bounded distance from the associated set $X(\rho)$. To fix attention on the underlying pants decomposition we adopt the notation

$$\rho(n) = Q_n.$$

As the ray $X(t)$ is half-infinite, the path Q_n has the property that for some component domain W, we have

$$\mathrm{diam}_W(\{Q_n\}_{n=0}^{\infty}) = \infty$$

and, moreover, that if this property holds for more than one component domain, then the corresponding two subsurfaces are complementary separated domains.

Since each domain W for ρ with this property carries a unique geodesic g_W, it also carries a corresponding boundary point λ in $\partial\mathcal{C}(W)$ to which g_W is asymptotic. Given ρ we call the union of such boundary points a *boundary lamination* for ρ. Note that such laminations are *purely irrational*: a boundary lamination contains no simple

closed curves. When this union is disconnected, we associate real weights to each component up to scale.

The boundary laminations are topologized as follows: given a sequence λ_n of boundary laminations, we say λ_n converges to λ if either

(1) there is a single domain W and a connected $\lambda \in \partial\mathcal{C}(W)$ for which $\pi_W(\lambda_n) \to \lambda$, or

(2) there are two complementary separated domains W and W^c in S for which $\pi_W(\lambda_n)$ converges to $\lambda_W \in \partial\mathcal{C}(W)$ and $\pi_{W^c}(\lambda_n)$ converges to $\lambda_{W^c} \in \partial\mathcal{C}(W^c)$ and

$$\lim_{n\to\infty} \frac{d_W(P,\lambda_n)}{d_{W^c}(P,\lambda_n)} = m$$

in which case we have

$$\lambda = m\lambda_W + \lambda_{W^c}$$

if $m \in (0,\infty)$,

$$\lambda = \lambda_W$$

if $m = \infty$ and

$$\lambda = \lambda_{W^c}$$

if $m = 0$.

Then we have the following.

Theorem 15 *Let S be a surface for which $\zeta(S) \leq 3$. Then the* CAT(0) *boundary of the Weil-Petersson metric is homeomorphic to the space of boundary laminations.*

Proof Given a ray $r = X(t)$ in the visual sphere at $X = X(0)$, let $\lambda(r)$ denote the associated boundary lamination for $X(t)$. We first show that this association is well-defined and injective.

From the discussion preceding the statement of the theorem, each infinite Weil-Petersson geodesic ray $X(t)$ determines either a connected geodesic lamination λ or a weighted sum of connected laminations λ_1 and λ_2.

In the case when we have the connected lamination λ, it is easy to see that any other geodesic ray $Y(t)$ that lies a bounded distance from $X(s(t))$ for some reparametrization $s(t)$, determines the same boundary lamination λ.

Injectivity. To see that there is a unique asymptote class of geodesic rays with associated lamination λ we assume first that the minimal subsurface $S(\lambda) = W$ containing λ is not a separated domain.

With this assumption, the first possibility is that $W = S$. Then λ is a filling lamination in the boundary of the curve complex of S. Now suppose $X_1(t)$ and $X_2(t)$ are geodesic rays through the base point x_0, each determining λ. Associated to $X_1(t)$ and $X_2(t)$ are hierarchy paths $\rho_1(n)$ and $\rho_2(n)$ each with some initial pants decomposition Q_0. The main geodesics m_1 and m_2 of ρ_1 and ρ_2 are infinite in the curve complex of S, and each converges to λ.

By the first property of hierarchies, for any subsurface W such that $d_W(Q_0, \lambda) \geq M_2$, there are connected intervals of times $[s_1, t_1]$ and $[s_2, t_2]$, such that for every time in these intervals the hierarchy paths ρ_1, ρ_2 contain ∂W. Furthermore by the fourth property of hierarchies, and the triangle inequality, since both hierarchies start with Q_0,

$$d_W(\rho_1(s_1), \rho_2(s_2)) \leq 2M_1,$$

and since both main geodesics converge to λ,

$$d_W(\rho_1(t_1), \rho_2(t_2)) \leq 2M_1.$$

This means that ρ_1 and ρ_2 are bounded distance apart at these times, and the same is then true of the corresponding Weil-Petersson geodesics X_1 and X_2.

Since the Weil-Petersson metric on the completion is CAT(0), the two geodesics stay a uniform distance apart between these two points. If there are infinitely many such W, this proves the geodesics determine the same asymptote class. If there are only finitely many, then after some point, the hierarchy paths are bounded distance apart, since their main geodesics are bounded distance apart and projections to all subsurfaces are bounded. The same is then true for the geodesics.

Next assume $W \subsetneq S$ and W is not separated. Then the boundary stratum in $\overline{\text{Teich}(S)}$ determined by ∂W is not a product, and is in fact isometric to Teich(W). Let σ denote the simplex in the curve complex corresponding to the curves in ∂W that are not in ∂S. The condition that λ is a boundary lamination for $X(t)$ implies that there is a hierarchy path $\rho(n)$ whose underlying pants decomposition we again denote by Q_n, so that ∂W represent curves in Q_n for all n sufficiently large. Thus $X(t)$ lies at a bounded distance from the boundary stratum Teich$_\sigma(S)$ corresponding to the vanishing of the extended length function ℓ_σ for the simplex σ. Since this stratum Teich$_\sigma(S)$ is a lower dimensional Teichmüller space ($\zeta(W) < 3$) we have that Teich$_\sigma(S)$ is itself

Gromov-hyperbolic by [BF]. Thus, the lamination λ determines a unique asymptote class of geodesics in $\mathrm{Teich}_\sigma(S)$, and hence in $\mathrm{Teich}(S)$.

The discussion when W is a separated domain follows from the limiting case when λ is disconnected.

When the associated boundary lamination is disconnected, and breaks into components λ_1 and λ_2, then there is a separating curve γ and a pair of complementary domains W_1 and W_2 in $S \setminus \gamma$ with $\lambda_i \in \partial\mathcal{C}(W_i)$.

In this case, the pants decompositions Q_n contain the curve γ for all n sufficiently large. The implication for $X(t)$ is that for all t sufficiently large the surface $X(t)$ has a shortest pants decomposition P_t that is a uniformly bounded distance from a pants decomposition $Q_n \in X_\gamma$. Since the pants graph is quasi-isometric to the Weil-Petersson metric, we have that points on the geodesic $X(t)$ lie at a uniformly bounded distance from the boundary stratum $\mathrm{Teich}_\gamma(S)$ where the extended length function ℓ_γ for γ vanishes.

The nearest point projection

$$\wp_\gamma \colon \mathrm{Teich}(S) \to \mathrm{Teich}_\gamma(S)$$

to the boundary stratum $\mathrm{Teich}_\gamma(S)$ determines a path

$$Z(t) = \wp_\gamma(X(t))$$

in $\mathrm{Teich}_\gamma(S)$. The projection of $Z(t)$ to each factor, in turn, lies a bounded distance from a unique geodesic. Let $g(t) \subset \mathrm{Teich}(W_1)$ denote the geodesic in the first factor and $h(t) \subset \mathrm{Teich}(W_2)$ the geodesic in the second.

Each infinite geodesic ray r based at $g(0) \times h(0)$ in the product

$$\{g(s) \times h(t) \mid s,t \in \mathbb{R}^+\}$$

is determined by its *slope*, in other words, the unique value of $m \in \mathbb{R}^+$ so that $r = \{(g(mt), h(t))\}$ where g and h are assumed parametrized by arclength.

Since the Weil-Petersson completion is CAT(0), it follows that $X(t)$ has a well defined *slope in* $\mathrm{Teich}_\gamma(S)$ namely, any geodesic in $\mathrm{Teich}_\gamma(S)$ within a uniformly bounded neighborhood of $X(t)$ has slope m. It follows that the boundary lamination

$$[m\lambda_1 + \lambda_2]$$

uniquely specifies the geodesic ray $X(t)$ based at $X(0)$.

For the final case when λ is connected but its minimal subsurface $S(\lambda)$ is a separated domain W, we note that this corresponds to the case above with slope $m = 0$ or $m = \infty$, in other words, one of the two factors is bounded.

Surjectivity. We now show that the assignment of a boundary lamination is surjective. In other words, we must show further that given any boundary lamination λ there is a geodesic ray with that boundary lamination associated to its asymptote class.

To see this for connected λ, we take a hierarchy path $\rho(n)$ whose only infinite geodesic lies in $\mathcal{C}(S(\lambda))$ and is asymptotic to λ. We denote by $C_n \in \overline{\text{Teich}(S)}$ the maximally noded surface obtained by pinching the curves in the underlying pants decomposition Q_n for $\rho(n)$.

Then the sequence of Weil-Petersson geodesic rays $X_n(t)$ beginning at $X = X_n(0)$, and terminating at the maximally noded surface C_n has a limit X_∞ in the visual sphere based at X, after passing to a subsequence (by compactness of the visual sphere).

We claim $X_\infty(t)$ is an infinite geodesic ray. Let

$$s_n = \frac{1}{\ell_X(Q_n)}$$

where $\ell_X(Q_n)$ denotes the total length of the geodesic representatives of the curves in Q_n on X. Then for each $X_n(t)$ we have

$$\ell_{X_n(t)}(s_n Q_n) \leq 1$$

by convexity of length functions along geodesics (see [Wol1]) since the length at X is 1, and the length converges to zero at the other endpoint.

Assume $X_\infty(t)$ has finite length T. Let μ be any limit of $s_n Q_n$ in $\mathcal{ML}(S)$ after passing to a subsequence.

Bicontinuity of length on $\text{Teich}(S) \times \mathcal{ML}(S)$ guarantees that for any $t < T$ we have

$$\ell_{X_\infty(t)}(\mu) \leq 1.$$

Let α be the (possibly empty) boundary of the minimal subsurface $S(\mu)$ containing μ. Let σ be the simplex in the curve complex corresponding so that $X_\infty(T) \in \text{Teich}_\sigma(S)$. We note that

$$\lim_{t \to T} \ell_{X_\infty(t)}(\mu) \leq 1$$

which guarantees that for each simple closed curve $\gamma \in \sigma^0$, we have

$$i(\gamma, \mu) = 0.$$

Otherwise, since $\ell_{X_\infty(t)}(\gamma) \to 0$ as $t \to T$, we would have $\ell_{X_\infty(t)}(\mu) \to \infty$. (In particular, we may conclude that μ does not fill S).

Noting that $X_n(T)$ converge in the completion to $X_\infty(T)$, we have that the distance d_n from $X_n(T)$ to the stratum $\text{Teich}_\sigma(S)$ is tending to zero. By the choice of Q_n, the

maximal cusps C_n lie at a bounded distance $D > 0$ from the σ-stratum $\mathrm{Teich}_\sigma(S)$ for all n, and their distance in the completion from $X_\infty(T)$ is diverging. This follows from the fact that the number of elementary moves from Q_n to a pants decomposition containing σ is uniformly bounded.

As the σ-stratum is totally geodesic, the fact that the completion $\overline{\mathrm{Teich}(S)}$ is a CAT(0) space guarantees that for any $T' > T$, the surfaces $X_n(T')$ converge into the σ-stratum as well. To see this note that the geodesics $Y_n(s)$ joining $X_\infty(T)$ to C_n have the property that for any s_0 the distance of $Y_n(s_0)$ from the σ-stratum is tending to zero. But the distance of the segments $X_n([T,T'])$ from the geodesics Y_n tends to zero, so the distance from $X_n(T')$ to $\mathrm{Teich}_\sigma(S)$ tends to zero.

It follows that the finite-length geodesic segments $X_n([0,T'])$ have endpoints converging in the completion and thus these limiting endpoints are the endpoints of a geodesic segment whose interior lies in the maximal stratum containing its endpoints (see [DW] and [Wol2]).

Since one endpoint of each geodesic $X_n([0,T'])$ is the base surface X, the interior of the limit segment lies in the interior of Teichmüller space. But by [Wol2], parametrizations of the approximating geodesics $X_n([0,T'])$ proportional to arclength converge to the parametrization proportional to arclength of the limit segment. Thus, the limit $X_\infty(T)$ of the sequence $\{X_n(T)\}$ lies in the interior of this geodesic limit, contradicting the assumption that $X_\infty(T) \in \mathrm{Teich}_\sigma(S)$. We conclude that the limiting geodesic X_∞ is infinite.

It follows that we can extract a limiting infinite geodesic ray X_∞ based at X, that lies within a uniform neighborhood of the union of maximal cusps $\cup_{n=0}^{\infty} C_n$. When λ is connected it follows that X_∞ has associated boundary lamination λ.

To treat the case when λ is disconnected, assume that

$$\lambda = m\lambda_1 + \lambda_2$$

where $\lambda_i \in \mathcal{C}(W_i)$ lies in the boundary of the curve complex of the separated domain W_i. Let γ be the separating curve for which $S \setminus \gamma = W_1 \sqcup W_2$. Then there is a geodesic g_m in

$$\mathrm{Teich}_\gamma(S) = \mathrm{Teich}(W_1) \times \mathrm{Teich}(W_2)$$

of slope m in the γ-stratum running through the nearest point X' to X in $\mathrm{Teich}_\gamma(S)$. Let $C_n \in \overline{\mathrm{Teich}(S)}$ denote a collection of points so that $C_0 \in \overline{\mathrm{Teich}_\gamma(S)}$ is the maximally noded surface closest to X', and

(1) each C_n is a maximally noded surface in $\overline{\mathrm{Teich}_\gamma(S)}$,

(2) each C_n lies a uniform distance D from the geodesic g_m, and

(3) the pants decompositions Q_n pinched in C_n determine a hierarchy path $\rho(n)$ for which the projection $\pi_{W_i}(\rho(n))$ is asymptotic to λ_i in $\partial\mathcal{C}(W_i)$.

Then we may apply the previous argument to conclude that the limit X_∞ of the geodesic segments $X_n(t)$ joining X to C_n is an infinite geodesic ray at X that lies uniformly bounded distance from the geodesic g_m; in other words, X_∞ lies in the asymptote class with projective boundary lamination

$$m\lambda_1 + \lambda_2.$$

Continuity. We now show that the assignment of a lamination to a ray is continuous. A family of infinite rays $X_n(t)$ based at $X(0)$ converge to $X(t)$ if there is a constant D so that for each $T > 0$ there is an n so that

$$d(X_n(t), X(t)) < D$$

for all $t \leq T$. By the quasi-isometry between the pants graph and the Weil-Petersson metric, we have a D' for which

$$d(P_n(t), P(t)) < D'$$

for all $t \leq T$, where $P_n(t)$ and $P(t)$ are shortest pants decompositions on $X_n(t)$ and $X(t)$ respectively (and are thus the images of $X_n(t)$ and $X(t)$ under the quasi-isometry).

But for any $W \subset S$, the above guarantees that for any $T > 0$ there is an N_T so that

$$d_W(P_n(T), P(T)) < D'' \tag{4.8}$$

for all $n > N_T$.

If $\lambda \in \partial\mathcal{C}(W)$ is a component of the boundary lamination for $X(t)$, then, we have that $\pi_W(P(t)) \to \lambda$ as $t \to \infty$. Thus, if λ_n is the boundary lamination for X_n, we have $\pi_W(\lambda_n) \to \lambda$, by an application of the fact that $\pi_W(P_n(T))$ lies a bounded distance from the geodesic in $\mathcal{C}(W)$ joining $\pi_W(P_n(0))$ to $\pi_W(\lambda_n)$ and that the bound 4.8 holds for each T and all $n > N_T$.

This suffices to show continuity in the case when λ is connected. For the disconnected case, we apply the bound 4.8 to each separated domain and observe that the divergence of the projections guarantees that the ratios converge.

Thus, the assignment of a boundary lamination to an asymptote class of geodesic rays is a homeomorphism, and the proof is complete.

□

5 Non-Relative-Hyperbolicity

In this section we address the question of the strong relative hyperbolicity of $P(S)$ and the Weil-Petersson metric on Teich(S) when $\zeta(S) > 3$. We will borrow extensively from the ideas and terminology of [BDM], who show that for surfaces S with $\zeta(S) \geq 6$ that $P(S)$ is thick, a condition which will, in this context, guarantee that $P(S)$ is not strongly relatively hyperbolic with respect to any collection of co-infinite subsets (a subset of a metric space is *co-infinite* if there are points in the space at an arbitrarily large distance from the set).

For the purposes of exposition, we say S is of *mid-range complexity* if $\zeta(S) \in \{4,5\}$, in other words, $S = S_{g,n}$ and we have

$$(g,n) \in \{(0,7),(0,8),(1,4),(1,5),(2,1),(2,2)\}.$$

Definition 16 *A curve γ is said to be* domain separating *if it separates S into two components Y and Y^c, neither of which is a 3 holed sphere.*

Accordingly, let $\mathcal{C}_{\text{sep}}(S) \subset \mathcal{C}(S)$ be the set of domain separating curves.

If $\gamma \in \mathcal{C}_{\text{sep}}(S)$ and Y_1 and Y_2 are disjoint subsurfaces of $S \setminus \gamma$ with $\zeta(Y_i) \geq 1$, $i = 1,2$, any two hierarchy paths ρ_1 and ρ_2 in $P(Y_1)$ and $P(Y_2)$ determine a *quasi-flat*

$$\rho_1 \times \rho_2 \colon \mathbb{Z} \times \mathbb{Z} \to P(S),$$

namely, quasi-isometrically embedding $\mathbb{Z} \times \mathbb{Z}$ in $P(S)$ with constants not depending on γ, ρ_1 or ρ_2 (this was observed in [BF] – it follows from the distance formula (2.1)).

In [BDM] a general definition is given for a collection of metric spaces to be *uniformly thick of order at most $n+1$*.

Definition 17 *([BDM] Definition 7.1) A metric space is* thick of order zero *if it is* unconstricted.

A metric space is thick of order at most $n+1$ with respect to a collection $\mathcal{L}$ of subsets of X if

- *with their restricted metric from X, the subsets in $\mathcal{L}$ are uniformly thick of order n,*
- *for some fixed $r > 0$,*

  $$X = \cup_{L \in \mathcal{L}} \mathcal{N}_r(L),$$

  and

- *any two elements L and L' in $\mathcal{L}$ can be thickly connected; there exists a sequence $L = L_1, L_2, \ldots, L_m = L'$ with $L_i \in \mathcal{L}$ and with*

$$\operatorname{diam}(\mathcal{N}_r(L_i) \cap \mathcal{N}_r(L_{i+1})) = \infty$$

for all $1 \leq i \leq m-1$.

A collection $\{X_i\}$ is called uniformly thick of order at most $n+1$ *if a uniform r can be taken in the above definition.*

The condition that a metric space be *unconstricted* makes use of the asymptotic cone of a metric space, which will not be necessary for our considerations. We will work instead with uniform quasi-flats discussed above, which are themselves *uniformly unconstricted* (see [BDM]).

Theorem 18 *If $\zeta(S) \in \{4,5\}$ and $S \neq S_{2,1}$ then $P(S)$ is thick of order 1. If $S = S_{2,1}$ then $P(S)$ is thick of order at most 2.*

In [BDM] the first statement of the theorem is established for $P(S)$ when $\zeta(S) \geq 6$, as well as for the mapping class group when $\zeta(S) \geq 2$.

As in what follows, their proof is based on finding thickly connected chains of quasi-flats. In our cases, the existence of such chains relies on a detailed study of the connectivity of the sub-complex of *domain separating* curves $\mathcal{C}_{\text{sep}}(S) \subset \mathcal{C}(S)$.

Lemma 6 *Let $S = S_{g,n}$, where $(g,n) \in \{(1,4),(1,5),(0,7),(0,8)\}$. Then $\mathcal{C}_{\text{sep}}(S)$ is connected.*

Proof Let α and β lie in $\mathcal{C}_{\text{sep}}(S)$. It suffices either to find $\gamma \in \mathcal{C}_{\text{sep}}(S)$ disjoint from α such that $i(\gamma,\beta) < i(\alpha,\beta)$, or to replace α with $\alpha' \in \mathcal{C}_{\text{sep}}(S)$ disjoint from α so that $i(\alpha',\beta) = i(\alpha,\beta)$ and then find such a γ. For then in at most two steps we have reduced intersection numbers, and inductively we can find the desired path.

We now consider the cases of $S_{1,4}$ and $S_{0,7}$. Let η be an arc in the complement of α joining β to the puncture p. Then we say β' is obtained from β by *moving the puncture p across β along η* if β' is the component of the boundary of a regular neighborhood of $\beta \cup \eta$ not isotopic to β.

Start with the case $S = S_{1,4}$. Given α and β in $\mathcal{C}_{\text{sep}}(S)$ we may move a puncture across one of these along an arc in the complement of the other if necessary to arrange that they enclose different numbers of punctures, either four or three. Without loss of generality assume that $S \setminus \alpha$ contains a subsurface Y containing four punctures.

We claim that not all arcs of $\beta \cap Y$ with endpoints on α can lie in a the homotopy class mod α that separates the four punctures in Y into two pairs of punctures for each pair lies in a single component of the complement of β one of which must contain three punctures by hypothesis. Thus there must be a homotopy class of arcs in Y that has exactly 3 punctures in its complement, and now a surgery produces a curve γ disjoint from α.

For $S = S_{0,7}$ every curve in $\mathcal{C}_{\text{sep}}(S)$ divides the surface into two components, one of which contains three punctures and one of which contains four punctures. Let Y be the component of $S \setminus \alpha$ that contains four punctures. If some arc of $\beta \cap Y$ separates one puncture from the other three, we may perform a surgery as above. Thus assume each arc of $\beta \cap Y$ separates the punctures in Y into two pairs of punctures. One pair lies in the component of the complement of β containing four punctures.

There are two cases. In the first case, all four of the punctures in Y are in the same component of the complement of β. In that case we can move any of the punctures across α, or equivalently find a disjoint α'. Then the three remaining punctures in the complement of α' are in the same component of the complement of β as the one moved puncture which is now in the component of the complement of α' containing four punctures. The other three punctures are in the other component of the complement of β, and thus the moved puncture is separated by an arc from those punctures. We have reduced to the case where we can now perform a surgery.

In the second case, a pair of punctures in Y are in the same component of the complement of β as a pair of punctures in the complement of Y. We now move one of these punctures in Y across α forming α'. Now there are three punctures in the component of the complement of α' that contains four punctures that are in the same component of the complement of β and again we can perform the surgery.

The cases $S = S_{0,8}$ and $S = S_{1,5}$ follow readily from the observation that filling in a puncture gives a well defined map from

$$\mathcal{C}_{\text{sep}}(S_{g,n}) \to \mathcal{C}(S_{g,n-1})$$

whose image lies in $\mathcal{N}_1(\mathcal{C}_{\text{sep}}(S_{g,n-1}))$ and contains $\mathcal{C}_{\text{sep}}(S_{g,n-1})$.

Thus, given a pair of curves in $\mathcal{C}_{\text{sep}}(S_{1,5})$ we may find separating curves at distance 1 from these whose images lie in $\mathcal{C}_{\text{sep}}(S_{1,4})$ after filling in the appropriate puncture, and similarly for $S_{0,8}$. These cases of the lemma then follow from the connectivity of $\mathcal{C}_{\text{sep}}(S_{1,4})$ and $\mathcal{C}_{\text{sep}}(S_{0,7})$.

□

Lemma 7 *Let $S = S_{2,2}$. Given $\gamma_1, \gamma_N \in \mathcal{C}_{\text{sep}}(S)$ there is a sequence $\gamma_1, \gamma_2, \ldots, \gamma_N$ of curves in $\mathcal{C}_{\text{sep}}(S)$ such that for each $i \leq N-1$, either γ_i and γ_{i+1} are disjoint or $S \setminus (\gamma_i \cup \gamma_{i+1})$ is a sphere with 4 punctures.*

Proof By filling in the punctures we can consider the curves as lying on a closed surface. By a result of Schleimer ([Sch], see also [Put]) on a closed surface of genus 2 there is a sequence $\gamma_1, \ldots, \gamma_N$ of separating curves such that successive curves intersect minimally, which means four times. This implies that some complementary component of their union is an annulus. Since on the punctured surface we may move the punctures across separating curves so that they both lie in one of these complementary annuli, we can produce a sequence of separating curves satisfying the conditions of the Lemma. □

Now suppose S is any surface of mid-range complexity other than $S_{2,1}$. If $Y \subset S$ is a proper essential subsurface with $\zeta(Y) \geq 1$ and $\rho(n)$ is a hierarchy path in $P(Y)$, we denote by $|\rho(n)|$ the collection of curves in the pants decomposition $\rho(n)$ together with components of ∂Y that are non-peripheral in S.

If Y_1 and Y_2 are disjoint essential subsurfaces of S and ρ_1 and ρ_2 are bi-infinite hierarchy paths in $P(Y_1)$ and $P(Y_2)$ for which the union $|\rho_1(n)| \cup |\rho_2(m)|$ (forgetting possible repetitions of curves) is a pants decomposition of S, then we denote by

$$\mathcal{Q}_{\rho_1,\rho_2} \colon \mathbb{Z} \times \mathbb{Z} \to P(S)$$

the natural quasi-flat determined by

$$\mathcal{Q}_{\rho_1,\rho_2}(m,n) = |\rho_1(n)| \cup |\rho_2(m)|.$$

Given $\gamma \in \mathcal{C}_{\text{sep}}(S)$, we let $\mathcal{L}_\gamma$ denote all such quasi-flats $\mathcal{Q}_{\rho_1,\rho_2}$ with image in X_γ (so that γ lies in each pants decomposition in the image of $\mathcal{Q}_{\rho_1,\rho_2}$). Finally, let $\mathcal{L}$ denote the union of all quasi-flats in all $\mathcal{L}_\gamma$, in other words

$$\mathcal{L} = \{\mathcal{Q} \in \mathcal{L}_\gamma : \gamma \in \mathcal{C}_{\text{sep}}(S)\}.$$

As we have remarked before, there are constants $K > 1$ and $C > 0$ such that each $\mathcal{Q}$ in $\mathcal{L}$ is (K, C)-quasi-isometrically embedded into $P(S)$, so the collection of quasi-flats $\mathcal{L}$ is uniformly thick of order 0, in the sense of [BDM].

Lemma 8 *If S has mid-range complexity and $S \neq S_{2,1}$ then every $P \in P(S)$ is within distance 1 of an element in $\mathcal{L}$.*

Proof Given a $\gamma \in \mathcal{C}_{\text{sep}}(S)$ and a pants decomposition P containing γ, we can find an element of $\mathcal{L}_\gamma$ containing P: this amounts to observing that there are bi-infinite hierarchy paths through any point in $P(Y)$ for Y a component of $S \setminus \gamma$. Thus, it suffices to show that each pants decomposition $P \in P(S)$ lies within distance 1 of some P' containing a separating curve.

In the case of the sphere it is obvious that every pants decomposition contains a curve in $\mathcal{C}_{\text{sep}}(S)$ so we consider the case of $S_{1,4}$. We can assume that there are curves β_1 and β_2 in P surrounding a pair of punctures each; otherwise we would be done. The complement of $\beta_1 \cup \beta_2$ is a torus Z with 2 punctures. If there is a curve in Z that bounds a punctured torus we again are done; so assume otherwise. This means that there are curves β_3 and β_4 in P which each cut Z into a 4 holed sphere. An elementary move now changes one of these into a pants decomposition containing a curve that bounds a punctured torus.

In the case of $S_{1,5}$ we can again assume the existence of β_1 and β_2 as in the previous case. The complement now is a torus Z with 3 holes. If there is a curve that surrounds both β_1 and β_2 or surrounds one of these curves and the remaining puncture x, we are finished. If there is a curve which cuts off a punctured torus, we are again finished. Thus, we assume the remaining possibility holds: there is a curve $\beta_3 \in P$ which cuts Z into a 5 holed sphere W. We can now assume there is a curve $\beta_4 \in P$ which cuts W into a 4 holed sphere V and a 3 holed sphere, and so that the union $\beta_3 \cup \beta_4$ separates Z. Without loss of generality we can assume β_1 is a boundary curve of V. Inside V there is a last curve $\beta_5 \in P$ which separates x from β_1; for otherwise we would be done. Now an elementary move inside V replaces β_5 with one that contains x and β_1, and hence lies in $\mathcal{C}_{\text{sep}}(S)$.

The case of $S_{2,2}$ is easier since in the closed genus 2 every pants decomposition is distance at most 1 from one containing a separating curve. □

The proof that $P(S)$, for $S \neq S_{2,1}$, is thick of order *at most* 1 is concluded by

Proposition 2 *Any two quasi-flats $\mathcal{Q}$ and $\mathcal{Q}'$ in $\mathcal{L}$ can be* thickly connected*: there exists a sequence*

$$\mathcal{Q} = \mathcal{Q}_1, \ldots, \mathcal{Q}_N = \mathcal{Q}'$$

with $\mathcal{Q}_i \in \mathcal{L}$, and for some fixed $r > 0$,

$$\operatorname{diam}(\mathcal{N}_r(\mathcal{Q}_i) \cap \mathcal{N}_r(\mathcal{Q}_{i+1})) = \infty$$

for all $1 \leq i \leq N-1$.

Proof Consider $S_{g,n}$ where $(g,n) \in \{(0,7),(0,8),(1,4),(1,5)\}$. The proof of the lemma in these cases follows from following two observations:

(1) Given $\gamma \in \mathcal{C}_{\text{sep}}(S)$, and quasi-flats $\mathcal{Q} = \mathcal{Q}_{\rho_1,\rho_2}$ and $\mathcal{Q}' = \mathcal{Q}_{\rho_1,\rho_2'}$ in $\mathcal{L}_\gamma$, there is a quasi-flat $\mathcal{Q}'' = \mathcal{Q}_{\rho_1,\rho_2''}$ in $\mathcal{L}_\gamma$ so that

$$\operatorname{diam}(\mathcal{Q} \cap \mathcal{Q}'') = \infty \quad \text{and} \quad \operatorname{diam}(\mathcal{Q}'' \cap \mathcal{Q}') = \infty.$$

(2) Given any disjoint pair γ and γ' in $\mathcal{C}_{\text{sep}}(S)$, there is a quasi-flat $\mathcal{Q}$ in $P(S)$ so that $\mathcal{Q} \in \mathcal{L}_\gamma \cap \mathcal{L}_{\gamma'}$.

To see the first statement, we need only observe that if Y is the component of $S \setminus \gamma$ containing $|\rho_2(0)|$ and $|\rho_2'(0)|$ there is a bi-infinite hierarchy path ρ_2'' in $P(Y)$ so that $\rho_2''(0) = \rho_2(0)$ and $\rho_2''(j) = \rho_2'(0)$ for some j. Then we have

$$\mathcal{Q}(\mathbb{Z} \times \{0\}) = \mathcal{Q}''(\mathbb{Z} \times \{0\}) \quad \text{and} \quad \mathcal{Q}''(\mathbb{Z} \times \{j\}) = \mathcal{Q}'(\mathbb{Z} \times \{0\}),$$

so each intersection $\mathcal{Q} \cap \mathcal{Q}''$ and $\mathcal{Q}'' \cap \mathcal{Q}'$ has infinite diameter.

To see the second assertion, observe that for disjoint curves γ and γ' in $\mathcal{C}_{\text{sep}}(S)$, that determine precisely two components Z_1 and Z_2 in $S \setminus \gamma \cup \gamma'$ with $\zeta(Z_i) \geq 1$ for $i = 1,2$, we may take ρ_i to be a bi-infinite hierarchy path in $P(Z_i)$, and the quasi-flat $\mathcal{Q}_{\rho_1,\rho_2}$ lies in $\mathcal{L}_\gamma \cap \mathcal{L}_{\gamma'}$, satisfying the claim. If $S \setminus \gamma \cup \gamma'$ has three components each with complexity 1, there is a third curve $\gamma'' \in \mathcal{C}_{\text{sep}}(S)$ disjoint from γ and γ' and separating them, so that taking Z_i to be the two components of $S \setminus \gamma \cup \gamma'$ not containing γ'' and ρ_1 and ρ_2 as before, we have a hierarchy path

$$\rho_2'(n) = |\rho_2(n)| \cup \gamma''$$

for which the quasi-flat $\mathcal{Q}_{\rho_1,\rho_2'}$ satisfies the claim.

When $S = S_{2,2}$ and $\mathcal{C}_{\text{sep}}(S)$ is not necessarily connected, we replace condition 2 with

$2'$ If $S = S_{2,2}$ and γ and γ' in $\mathcal{C}_{\text{sep}}(S)$ intersect in such a way that $S \setminus \gamma \cup \gamma'$ contains a 4-holed sphere Z, then there is an $r > 0$ and quasi-flats $\mathcal{Q} \in \mathcal{L}_\gamma$ and $\mathcal{Q}' \in \mathcal{L}_{\gamma'}$ for which

$$\operatorname{diam}(\mathcal{N}_r(\mathcal{Q}) \cap \mathcal{N}_r(\mathcal{Q}')) = \infty.$$

To see this, let Y be the component of $S \setminus \gamma$ not containing Z and let Y' be the component of $S \setminus \gamma'$ not containing Z. Let ρ_1 be any bi-infinite hierarchy path in $P(Z)$. Taking ρ_2 in $P(Y)$ and ρ_2' in $P(Y')$ to be bi-infinite hierarchy paths, we obtain quasi-flats

$$\mathcal{Q} = \mathcal{Q}_{\rho_1,\rho_2} \quad \text{and} \quad \mathcal{Q}' = \mathcal{Q}_{\rho_1,\rho_2'}$$

Then letting r be the distance between $\mathcal{Q}(0,0)$ and $\mathcal{Q}'(0,0)$, we see that

$$\operatorname{diam}(\mathcal{N}_r(\mathcal{Q}(\mathbb{Z}\times\{0\}))\cap\mathcal{N}_r(\mathcal{Q}'(\mathbb{Z}\times\{0\})))=\infty.$$

To prove the claim, we need only observe that the configuration of γ and γ' on S is unique up to homeomorphisms of S to see that r can be taken independently of the pair of curves γ and γ'.

Applying Lemma 7, we may find a sequence of curves $\{\gamma_i\}$ in $\mathcal{C}_{\text{sep}}(S)$ satisfying the conclusions of the lemma. Conditions 1, 2 and $2'$ guarantee that we can thickly connect quasi-flats in each $\mathcal{L}_{\gamma_i}$ to join quasi-flats in $\mathcal{L}_{\gamma_i}$ that thickly connect to quasi-flats in $\mathcal{L}_{\gamma_{i-1}}$ and $\mathcal{L}_{\gamma_{i+1}}$. The Proposition follows.

□

In [Behr] it was shown that $P(S)$ is not *unconstricted*, which, in particular, guarantees that $P(S)$ cannot be thick of order zero. It follows that in the cases above that $P(S)$ is thick of order *exactly* 1 (we thank Jason Behrstock for alerting us to this point).

We now consider the remaining case.

Proposition 3 *Let $S'=S_{2,1}$. The pants graph $P(S')$ is thick of order at most 2.*

Proof Let x denote the puncture on S' and let S be the closed surface of genus 2 obtained by adding x to S'. It is well-known that there exists an injective homomorphism

$$\pi_1(S,x)\to\operatorname{Mod}(S')$$

obtained by "pushing x around a loop." By a theorem of Kra (see [Kra]), if $\gamma\in\pi_1(S,x)$ has positive geometric intersection with every essential non-peripheral simple closed curve (γ is *filling*), then its image under this homomorphism is a pseudo-Anosov diffeomorphism. Let G be the image of $\pi_1(S,x)$.

Filling in the puncture also induces a map

$$\Pi:\mathcal{C}(S')\to\mathcal{C}(S).$$

The action of G preserves each fiber $\Pi^{-1}(\alpha);\alpha\in\mathcal{C}(S)$. It is known [Sch] (Proposition 4.3) that the fibers $\Pi^{-1}(\alpha)$ are connected. Given $\gamma\in\mathcal{C}(S)$, let $\widehat{X}_\gamma$ be the collection of pants decompositions in $P(S')$ so that each $P\in\widehat{X}_\gamma$ contains some $\gamma'\in\Pi^{-1}(\gamma)$.

Lemma 9 *Let $\gamma\in\mathcal{C}_{\text{sep}}(S)$. Then $\widehat{X}_\gamma$ is thick of order at most 1 and the collection $\mathcal{L}$ of all $\widehat{X}_\gamma$ for $\gamma\in\mathcal{C}_{\text{sep}}(S)$ is uniformly thick.*

Proof Each $\gamma' \in \Pi^{-1}(\gamma)$ divides the surface S' into subsurfaces Y_1 and Y_2 with $\zeta(Y_i) \geq 1$. Taking hierarchy paths ρ_i in $P(Y_i)$ and quasi-flats $\mathcal{Q}_{\rho_1,\rho_2}$ as before, connectedness of the fiber allows us to argue using conditions 1 and 2 from the previous Proposition that for any two curves γ' and γ'' in the fiber $\Pi^{-1}(\gamma)$, quasi-flats in $\mathcal{L}_{\gamma'}$ and $\mathcal{L}_{\gamma''}$ can be thickly connected within $\widehat{X}_\gamma$.

Since each $P \in \widehat{X}_\gamma$ lies in some quasi-flat of this form, we have $\widehat{X}_\gamma$ is thick of order 1. Since the constants do not depend on γ, the union $\mathcal{L}$ is a collection of uniformly thick subsets of $P(S')$. □

We now conclude the proof that $P(S')$ is thick of order at most 2.

Exactly as in the case of $S_{2,2}$, any point in $P(S')$ is within distance at 1 of a point in $\mathcal{L}$. Now we show that any two elements of $\mathcal{L}$ can be thickly connected. Given γ_1 and γ_N in $\mathcal{C}_{\text{sep}}(S')$, we join them by a sequence $\gamma_1, \ldots, \gamma_N$ in $\mathcal{C}_{\text{sep}}(S')$ where successive curves are either disjoint or intersect minimally. There is a uniform constant C' and pants decompositions, P_i that contain γ_i, such that

$$d(P_i, P_{i+1}) \leq C'.$$

Now let ϕ be a pseudo-Anosov element in G. The orbit

$$\{\phi^n(P_i)\}_{n=-\infty}^{\infty}$$

is an infinite diameter subset of $\widehat{X}_{\gamma_i}$. As ϕ acts isometrically on $P(S')$, we have

$$\begin{aligned} d(\phi^n(P_i), \phi^n(P_{i+1})) &= d(P_i, P_{i+1}) \\ &\leq C'. \end{aligned}$$

This guarantees that $\widehat{X}_{\gamma_1}$ and $\widehat{X}_{\gamma_N}$ can be thickly connected.

□

Proof of Theorem 6 By Theorem 18 (for $\zeta(S) = 4$ and 5) and [BDM, Cor. 7.9] the pants graph cannot be *asymptotically tree-graded* for each S with $\zeta(S) \geq 4$.

The theorem then follows immediately from the equivalence of strong relative-hyperbolicity with the condition that a metric space is asymptotically tree-graded (see [DS, Thm. 4.1]).

□

Remark: It is interesting to note that the proof of Theorem 6 does not show that $P(S_{2,1})$ is thick of order exactly 2. It would be interesting to know whether $P(S_{2,1})$ presents such a special case.

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Department of Mathematics, Brown University, Box 1917, Providence, RI 02912
Department of Mathematics, UIC, Chicago, IL 60607
brock@math.brown.edu, masur@math.uic.edu
http://www.math.brown.edu/~brock, http://www.math.uic.edu/~masur