Pants decompositions and the Weil-Petersson metric

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1 Introduction

Since notions of coarse geometry and quasi-isometries were first introduced by M. Gromov, many studies of geometry have been renovated with its rough perspective. In this note we give an expository account of results of [Br] and of joint work of the author with Benson Farb [BF] that apply such a coarse point of view to the Weil-Petersson metric on Teichmüller space.

A natural graph of pants decompositions of surfaces, introduced by A. Hatcher and W. Thurston, provides an organizing combinatorial structure for the Weil-Petersson metric. The coarse structure of this graph provides a new tool in the geometric study of Teichmüller spaces.

We briefly introduce the two objects to be compared.

Pants decompositions. A pants decomposition of a closed surface S is a decomposition of the surface along simple closed curves into three-holed spheres (the *pants* of the pants decomposition). Such a decomposition is determined by a choice of a maximal collection P of simple closed curves on S so that

1. elements of P are pairwise disjoint,
2. each element α of P is homotopically essential and non-peripheral (α is not boundary-parallel),

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3. no two elements of P are freely homotopic.

Though elements of P represent the boundary curves of the three-holed spheres in a pants decomposition, a standard locution is to refer to the collection of curves P itself as a pants decomposition of S.

By organizing all pants decompositions of S as the vertices of a simplicial complex, Hatcher and Thurston introduced a useful combinatorial tool to understand pants decompositions on surfaces. In their construction, pants decompositions P and P' are related by an *elementary move* if P' can be obtained from P by removing a single curve $\alpha \in P$ and replacing it with a curve β satisfying the conditions

1. $P' = (P \setminus \alpha) \cup \beta$ is a pants decomposition,
2. β has minimal intersection number $i(\alpha, \beta)$ with α among all curves satisfying (1).

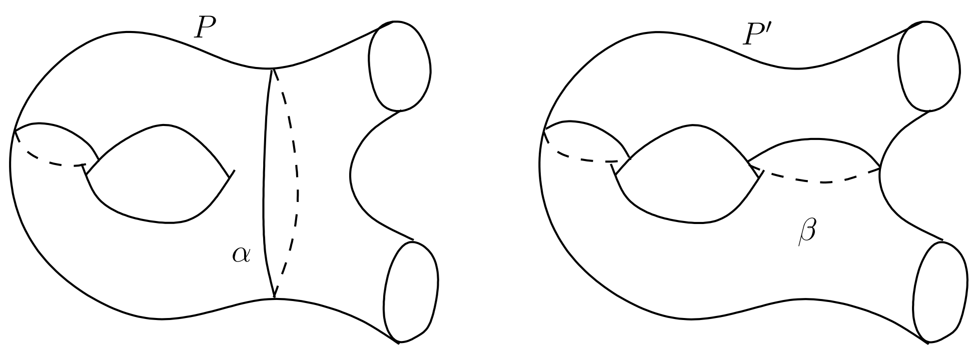

Figure 1. An elementary move on pants decompositions.

Here, the *intersection number* $i(\alpha, \beta)$ is the minimal number of intersection points of representatives of the isotopy classes of α and β.

The notion of an elementary move collates the pants decompositions of S:

Definition 1.1 *The* pants graph $\mathbf{P}(S)$ *is the graph whose vertices are pants decompositions of* S *and whose edges join vertices whose corresponding pants decompositions differ by an elementary move.*

As a first study of their graph, Hatcher and Thurston established that $\mathbf{P}(S)$ is connected. In fact, they show that certain canonical two cells can be added to obtain a simply connected 2-complex $C_{\mathbf{P}}(S)$ which we will call the *pants complex* [HLS] [HT].

Assigning each edge of $\mathbf{P}(S)$ length 1, we obtain a natural distance function

$$d_{\mathbf{P}}(.,.) \colon \mathbf{P}^0(S) \times \mathbf{P}^0(S) \to \mathbb{N}$$

on the vertices $\mathbf{P}^0(S)$. The function $d_{\mathbf{P}}(P, P')$ measures the length of a minimal length path in $\mathbf{P}(S)$ joining P to P'.

We will discuss how the structure of the pants graph gives insight into the study of *Teichmüller space*.

The Weil-Petersson metric. Given a compact surface S of negative Euler characteristic, the Teichmüller space, $\mathrm{Teich}(S)$, is the collection of all finite area hyperbolic surfaces X equipped with homeomorphisms $f \colon \mathrm{int}(S) \to X$ with the equivalence relation

$$(f \colon \mathrm{int}(S) \to X) \sim (g \colon \mathrm{int}(S) \to Y)$$

if there is an isometry $\phi \colon X \to Y$ so that $\phi \circ f$ is isotopic to g. L. Ahlfors identified a natural complex structure on $\mathrm{Teich}(S)$.

Each such hyperbolic surface X is naturally a Riemann surface via its Fuchsian uniformization $X = \mathbb{H}^2/\Gamma$ where Γ is a *Fuchsian group*, i.e. a torsion-free discrete subgroup of $\mathrm{PSL}_2(\mathbb{R})$ acting by isometries on the upper-half-plane $\mathbb{H}^2 = \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$ with the *Poincaré metric*

$$ds^2 = \frac{dz^2}{\mathrm{Im}(z)^2}.$$

The *holomorphic quadratic differentials* $Q(X)$ on X are the holomorphic differential forms of type $\phi(z)dz^2$ (elements of $Q(X)$ are a holomorphic tensors of type$(2,0)$; see, e.g., [Gard]). Then $Q(X)$ is naturally identified with the holomorphic cotangent space $T^*_X(\mathrm{Teich}(S))$ to Teichmüller space at X.

Let $ds = \rho(z)|dz|$ denote the line-element for the Poincaré metric on X. The Weil-Petersson metric on $\mathrm{Teich}(S)$ arises from the hyperbolic L^2-norm

$$\|\varphi\|_{\mathrm{WP}} = \left(\int_X \frac{|\varphi|^2}{\rho^2} dz d\overline{z} \right)^{\frac{1}{2}}$$

on $Q(X)$ via polarization using the usual pairing $\langle \mu, \varphi \rangle_X = \int_X \mu\phi |dz|^2$ between the tangent vector $\mu \in T_X(\text{Teich}(S))$ (an infinitesimal Beltrami differential) and the cotangent vector φ.

Many interesting properties of the Weil-Petersson metric on Teichmüller space have emerged from work of Ahlfors, Wolpert, Masur, Tromba and others. We will be interested in the Riemannian part g_{WP} and its associated distance function $d_{\text{WP}}(.,.)$ from now on. It is known (see [Tro], [Wol2]) that the Weil-Petersson metric has negative sectional curvature, but that the sectional curvatures are bounded neither away from zero nor infinity.

In [Wol3], Wolpert shows that g_{WP} is geodesically convex, but g_{WP} is not complete [Wol1]. Masur proved that its completion gives a natural metric on the *augmented Teichmüller space*, obtained by adjoining noded Riemann surfaces at the boundary, where certain collections of simple closed curves become pinched to cusps (see [Mas], [Bers]).

Correspondences. In [Br], we add the following large-scale property to the above list of properties that distinguish the Weil-Petersson metric.

Theorem 1.2 ([Br, Thm 1.1]) *The Weil-Petersson metric g_{WP} on the Teichmüller space of S is quasi-isometric to the graph $\mathbf{P}(S)$.*

The theorem asserts the existence of a mapping $Q\colon \mathbf{P}(S) \to \text{Teich}(S)$ and constants $K_1 > 1$, K_2, $K_3 > 0$ so that

$$\frac{d_{\mathbf{P}}(P_1, P_2)}{K_1} - K_2 \leq d_{\text{WP}}(Q(P_1), Q(P_2)) \leq K_1 d_{\mathbf{P}}(P_1, P_2) + K_2$$

whose image is K_3 dense.

The theorem is quite similar in spirit to a theorem of Masur and Minsky [MM1]:

Theorem 1.3 (Masur-Minsky) *The* electric Teichmüller space *is quasi-isometric to the complex of curves.*

The *complex of curves* $\mathcal{C}(S)$ is a simplicial complex whose zero skeleton contains a vertex for each isotopy class of essential, non-peripheral simple closed curves on S, and whose k-simplices span collections of $k+1$ vertices whose corresponding curves can be realized disjointly on S. The space $\mathcal{C}(S)$ is connected, so making each simplex a standard Euclidean simplex yields a metric on $\mathcal{C}(S)$. This metric gives some kind of measure of the complexity of a given curve relative to another. The curve complex, then, models a kind of quotient of the Teichmüller space with its Teichmüller metric, where the regions

$$H_\alpha = \{X \in \mathrm{Teich}(S) \mid \ell_X(\alpha) < \epsilon\}$$

are crushed to have diameter 1. This *electric Teichmüller space* is then shown to be quasi-isometric to $\mathcal{C}(S)$ (the regions H_α have been *electrified* in the sense of [Fa]).

The parallel with theorem 1.2 lies in the fact that each pants decomposition of S is naturally associated to a region of bounded Weil-Petersson diameter in $\mathrm{Teich}(S)$, without no electrification (see section 2).

The utility of theorem 1.2 can be seen in the following theorem, which answers a question of Brian Bowditch [Be].

Theorem 1.4 (B-Farb) *The Weil-Petersson metric g_{WP} on* $\mathrm{Teich}(S)$ *is Gromov-hyperbolic if and only if S is a four or five-holed sphere or a one or two-holed torus.*

The condition of Gromov-hyperbolicity for a path metric space X entails the existence of some $\delta > 0$ for which the space X is *δ-hyperbolic*. In a δ-hyperbolic space, each geodesic triangle T in X is δ-thin: each side of T is contained in the δ neighborhood of the union of the other two sides. (The space X is called *Gromov-hyperbolic* when no reference to the specific hyperbolicity constant δ is necessary). Clearly, δ-hyperbolicity is a quasi-isometry invariant of a path-metric space. This notion of coarse negative curvature has had myriad applications in the study of groups, rigidity, and low-dimensional topology.

As in the case of theorem 1.2, theorem 1.4 is inspired by a result of Masur and Minsky [MM1].

Theorem 1.5 (Masur-Minsky) *The complex of curves $\mathcal{C}(S)$ is Gromov-hyperbolic.*

That the Weil-Petersson metric is not Gromov-hyperbolic for higher dimensional Teichmüller spaces points to the way in which the pants complex differs from the curve complex. In the pants complex, one can make independent elementary moves on disjoint subsurfaces to generate flat subspaces (see section 4). Such subspaces are natural obstructions to δ-hyperbolicity. The instances when the Weil-Petersson metric is Gromov-hyperbolic correspond precisely to the situations in which no such collection of independent elementary moves is possible, due to the lack of available subsurfaces on which to perform them.

In this paper, we will give an informal expository discussion theorems 1.2 and theorem 1.4 in the interest of conveying the main ideas. Detailed arguments may be found in [Br] and [BF]. Section 2 gives a detailed discussion the proof of theorem 1.2 when S is a one-holed torus. Section 3 discusses the role of *relative* hyperbolicity in the proof of theorem 1.4, and section 4 discusses how naturally arising quasi-flats in $\mathbf{P}(S)$ give an obstruction to hyperbolicity in the higher genus cases. We conclude the paper with a list of questions.

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2 Modeling Teichmüller space

We briefly explain the idea of the proof of theorem 1.2, treating the case when S is a one-holed torus. Note that in this case, $\mathbf{P}(S)$ is simply the Farey graph, and a pants decomposition P of S consists of a single essential, non-peripheral simple closed curve up to isotopy. Pants decompositions P and P' are related by an elementary move if their corresponding curves have intersection number 1 on S.

Since a pants decomposition is just a single simple closed curve up to isotopy, we will refer to such isotopy classes of curves as associated to vertices of $\mathbf{P}(S)$.

1. Applying Wolpert's non-completeness theorem and Masur's description of the completion ([Wol1, Mas]) given $\ell > 0$, each $\alpha \in \mathbf{P}^0(S)$ determines a bounded diameter region

   $$V_\ell(\alpha) = \{X \in \mathrm{Teich}(S) \mid \ell_X(\alpha) < \ell\}.$$

   Each point in X is a uniformly bounded distance from the node $N_\alpha \in \overline{\mathrm{Teich}(S)}$ in the completion of the Weil-Petersson metric where α is pinched to a cusp (see [Bers], [IT]).

2. By a theorem of Bers, there is a choice of $\ell = L$ so that the union of regions

   $$\bigcup_{\alpha \in \mathbf{P}^0(S)} V_L(\alpha)$$

covers Teichmüller space. For some extra security, we let

$$V(\alpha) = V_{2L}(\alpha).$$

3. We define a mapping $Q\colon \mathbf{P}^0(S) \to \mathrm{Teich}(S)$ simply by choosing any point $Q(\alpha)$ in $V(\alpha)$.

We remark briefly on part (1), which is critical to theorem 1.2. In [Wol1] and [Wol2], it is demonstrated that the non-completeness of the Weil-Petersson metric corresponds precisely to the *pinching deformation*, a smooth path $X_t \subset \mathrm{Teich}(S)$ on which the length $\ell_{X_t}(\alpha)$ of a simple closed geodesic α on X_t tends to zero. Wolpert estimates that given $\ell > 0$ there is a $C > 1$ so that given $X_0 \in \mathrm{Teich}(S)$ for which $\ell_{X_0}(\alpha) < \ell$, there is a pinching deformation X_t, $t \in [0,1]$, whose total Weil-Petersson length is less than $C\sqrt{\ell_{X_0}(\alpha)}$.

Since there is a unique noded surface N_α in the Weil-Petersson completion (since $S \setminus \alpha$ is a three-holed sphere), any two surfaces X and X' in $V(\alpha)$ have distance from N_α uniformly bounded in terms of L. Geodesic convexity implies that $d_{\mathrm{WP}}(X, X')$ is uniformly bounded.

The mapping Q is clearly D-dense where $D = \mathrm{diam}_{\mathrm{WP}}(V(\alpha))$. One must then show that the combinatorics of elementary moves in $\mathbf{P}(S)$ is reflected in the arrangements of the regions $V(\alpha)$.

First, a relatively simple argument shows that the mapping Q is $2D$-Lipschitz: for in this case, an elementary move $\alpha \to \alpha'$ corresponds to a pair of simple closed curves with intersection number one. The symmetric hyperbolic punctured torus $X \in \mathrm{Teich}(S)$ on which α and α' are geodesics meeting orthogonally and of minimal length over all curves in $\mathbf{P}^0(S)$ must lie in $V(\alpha)$ and $V(\alpha')$ (see figure 2).

It follows that the regions $V(\alpha)$ and $V(\alpha')$ have non-empty intersection, so points at distance 1 in $\mathbf{P}(S)$ map to points at distance at most $2D$ in $\mathrm{Teich}(S)$.

To see that Q does not contract distance too much, we prove 2 facts:

I. There is a $B > 0$ depending only on S, so that given X in $\mathrm{Teich}(S)$ for which X lies in $V(\alpha) \cap V(\alpha')$, the distance between α and α' in $\mathbf{P}^0(S)$ is bounded by B.

II. There is a uniform J so that for any geodesic segment g in $\mathrm{Teich}(S)$ of length 1 in the Weil-Petersson metric, there are curves $\alpha_1, \ldots, \alpha_k$, in $\mathbf{P}^0(S)$, $k < J$, so that the regions $\cup_i V(\alpha_i)$ cover g.

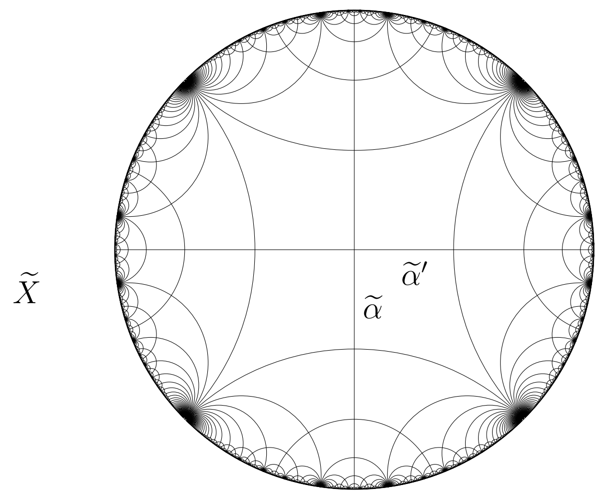

Figure 2. The tiling of $\mathbb{H}^2$ by the fundamental domain for the square torus X. The horizontal and vertical lines are the lifts of α and α'.

First, let us see that (**I**) and (**II**) suffice. By (**II**) we can cover a geodesic segment g in Teich(S) of length d with a collection of regions $V(\alpha_1) \cup \ldots \cup V(\alpha_l)$ where l is bounded by a uniform multiple of d. Applying (**I**), two overlapping sets $V(\alpha)$ and $V(\alpha')$ correspond to curves α and α' at uniformly bounded distance in $\mathbf{P}(S)$.

Thus, if $V(\alpha_I)$ contains the initial point of g and $V(\alpha_T)$ contains the terminal point of g, the distance from α_I to α_T in $\mathbf{P}(S)$ is bounded by a uniform multiple of d.

To see that (**I**) holds, note that by the collar lemma [Bus, Thm. 4.4.6], if X lies in $V(\alpha)$ then there is a uniform $\epsilon > 0$ so that the geodesic representative α^* of α on X has an embedded metric ϵ-collar $\mathcal{N}_\alpha$ on X. For any β for which $X \in V(\beta)$, the geodesic representative β^* has uniformly bounded intersection with α^* since $\ell_X(\beta) < 2L$ and each intersection accounts for a unique segment of $\beta^* \cap \mathcal{N}_\alpha$ of length 2ϵ. By an elementary topological argument one can see that up to the action of Dehn-twists about α, there are only finitely many curves $\beta_1, \ldots, \beta_m$ for which $X \in V(\alpha) \cap V(\beta_j)$, where $j = 1, \ldots, m$. But Dehn-twisting β about α does not change its distance

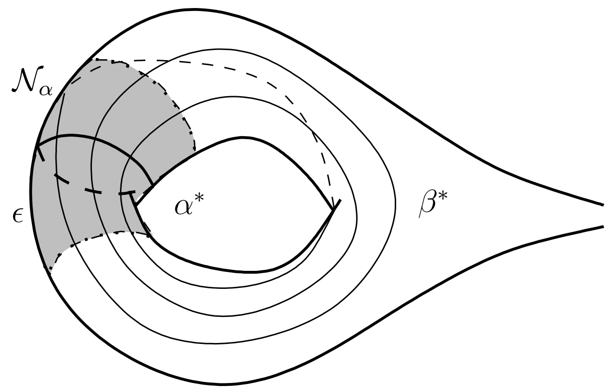

Figure 3. Each crossing of α^* requires definite length on β^*.

from α in $\mathbf{P}(S)$. Taking the maximum over j of $d_{\mathbf{P}}(\alpha, \beta_j)$ gives the desired bound.

To see that (**II**) holds, recall we have chosen $L > 0$ so that

$$\bigcup_{\alpha \in \mathbf{P}^0(S)} V_L(\alpha)$$

covers Teich(S) and so that $V(\alpha) = V_{2L}(\alpha)$. For any ℓ, the boundary $\partial V_\ell(\alpha)$ is the set of $X \in$ Teich(S) for which $\ell_X(\alpha) = \ell$. Appealing to Fenchel-Nielsen coordinates (see [IT]) the set $\partial V_\ell(\alpha)$ is parameterized by the *twisting* about α, so $\partial V_\ell(\alpha)$ is a copy of $\mathbb{R}$. If τ_α is a Dehn-twist about α, then τ_α acts on this copy of $\mathbb{R}$ by 2π-translations. Thus the set

$$\partial V_\ell(\alpha)/\langle \tau_\alpha \rangle$$

is a circle for each ℓ, and is in particular compact.

Since τ_α acts on Teich(S) by isometries in the Weil-Petersson metric, it follows that the Weil-Petersson distance from $\partial V(\alpha)$ to $\partial V_L(\alpha)$ is uniformly bounded *below* by some ϵ_0. Since general mapping classes also act by isometries, and the action of the mapping class group is transitive on $\mathbf{P}^0(S)$, the same ϵ_0 holds for all $\alpha \in \mathbf{P}^0(S)$.

Choose $\{\alpha_j\}$ so that

$$\bigcup_{\alpha_j} V_L(\alpha_j)$$

cover the unit length geodesic segment u. Since for each $\ell > 0$, the set $V_\ell(\alpha_j)$ is geodesically convex (see [Wol3]) we may conclude that the intersection

$$u_j = u \cap V(\alpha_j)$$

is an interval of width at least ϵ_0. Thus, if J is the smallest integer larger than $1/\epsilon_0$ we may pass to a subcollection $\{\alpha_i\}_{i=1}^k$ with $k < J$ so that

$$\bigcup_{i=1}^{k} V(\alpha_i)$$

covers u.

This completes the proof of assertions (**I**) and (**II**) for the case when S is a one-holed torus. The general case is only combinatorially more complicated.

3 Coarse curvature

In light of theorem 1.2, the model $\mathbf{P}(S)$ may be exploited to understand the coarse geometry of Teich(S) with the Weil-Petersson metric. In particular, combinatorics of $\mathbf{P}(S)$ yield the following dichotomy for the coarse curvature of the Weil-Petersson metric, which we recapitulate from section 1 (see [BF]).

Theorem 3.1 (B-Farb) *The Weil-Petersson metric g_{WP} on* Teich(S) *is Gromov-hyperbolic if and only if S is a four or five-holed sphere or a one or two-holed torus.*

The cases when S is a one-holed torus or four-holed sphere are familiar: in this case the graph $\mathbf{P}(S)$ is the *Farey graph* which is known to be Gromov-hyperbolic (see [Min] for a nice proof of this fact).

The cases when S is a two-holed torus or five-holed sphere represent interesting sporadic cases. We note that, in particular, the *Teichmüller metric* is *not* Gromov-hyperbolic in these cases (see [MW1]). What forces hyperbolicity in this case?

The notion of *relative hyperbolicity*, introduced by B. Farb [Fa], applies to spaces X that are Gromov-hyperbolic when some collection of subsets are crushed to have diameter 1. Farb accomplishes the crushing, or *electrifying*, of a subset $H \subset X$ by adding a single point p_H to the space X and joining each point in H to p_H by a geodesic segment of length 1/2.

Common examples arise in consideration of fundamental groups of hyperbolic manifolds with cusps: higher-rank cusps prevent such groups from being hyperbolic, but one may, in effect, crush the corresponding abelian subgroups to have diameter 1 in the Cayley graph and obtain a Gromov-hyperbolic space relative to this procedure.

An isoperimetric argument, (see [Fa]) guarantees that when a metric space X is hyperbolic relative to some collection of subsets H_i each of which is *itself* uniformly δ-hyperbolic, there is a condition on geodesics which guarantees hyperbolicity of the original space X. This condition, referred to as the *bounded region penetration property*, (or *bounded coset penetration property* in the context of groups) requires that geodesics that enter and exit a given region H_i near one another must fellow travel along the interior of the region.

In our setting, when S is a two-holed torus or five-holed sphere, the subgraph G_α of $\mathbf{P}(S)$ consisting of pants decompositions that contain a fixed curve α is itself the Farey graph of a component T of $S \setminus \alpha$, and is therefore hyperbolic. To prove hyperbolicity of $\mathbf{P}(S)$ for these cases, we show that

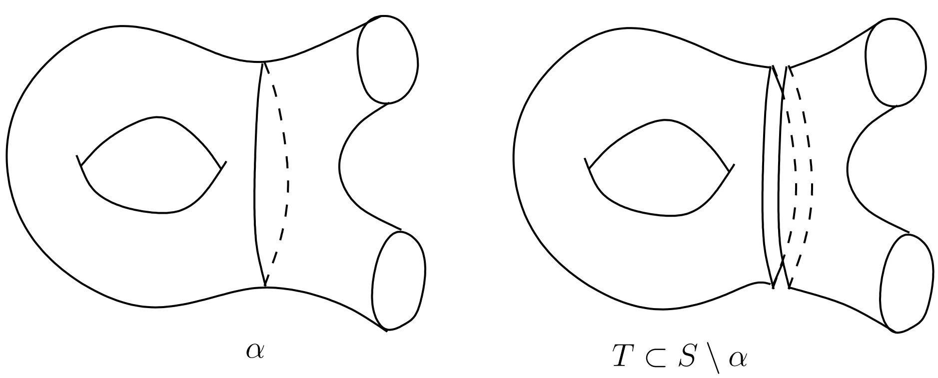

Figure 4. The pants decompositions containing α form a Farey graph.

the space $\mathbf{P}(S)$ is Gromov-hyperbolic *relative to the sets* G_α where α ranges over every isotopy class of simple closed curves on S.

To show this, we prove that relative to the collection of subsets G_α (after electrifying these subsets) the graph $\mathbf{P}(S)$ is quasi-isometric to the curve complex $\mathcal{C}(S)$. Relative hyperbolicity of $\mathbf{P}(S)$ then follows from the hyperbolicity theorem of Masur and Minsky (theorem 1.5).

Techniques from the study of hierarchies of geodesics in the *curve complex* [MM2] demonstrate that the pair $(\mathbf{P}(S), G_\alpha)$ of $\mathbf{P}(S)$ with its collection of subsets G_α satisfies the bounded region penetration property, so the original space $\mathbf{P}(S)$ is Gromov-hyperbolic in this case.

4 Quasi-flats in P(S)

Theorem 1.4 also guarantees that when S is not a one or two-holed torus or four or five-holed sphere then $\mathbf{P}(S)$ is *not* Gromov-hyperbolic.

To see why this is so, we recall a well known obstruction to Gromov-hyperbolicity of a space.

Definition 4.1 *Let X be a path metric space. An n-dimensional quasi flat F is a quasi-isometric mapping $F \colon \mathbb{R}^n \to X$.*

We show that when S is not a one or two-holed torus or a four or five-holed sphere then $\mathbf{P}(S)$ admits an n-dimensional quasi-flat for $n \geq 2$. The existence of such a quasi-flat is an immediate obstruction to Gromov-hyperbolicity, since $\mathbb{R}^2$ contains triangles violating the δ-thin condition for any δ. Since a 2-dimensional quasi-flat is the quasi-isometric image of $\mathbb{R}^2$, any space admitting a quasi-flat will possess triangles that violate the δ-thin condition for any δ.

Indeed, we prove the following [BF].

Theorem 4.2 (B-Farb) *Let S be a surface of genus g with p boundary components. Let $d(S)$ be the smallest integer such that $d(S) \geq (3g-3+p)/2$. Then $\mathbf{P}(S)$ admits a $d(S)$-dimensional quasi-flat.*

One easily sees that the only surfaces S for which $d(S) = 1$ are the one and two-holed torus and the four and five-holed sphere.

To see why such quasi-flats exist, we note that the import of the the integer $d(S)$ is the following: each surface S admits a decomposition into $d(S)$ subsurfaces each of which is either a one-holed torus or four-holed sphere, leaving an additional three-holed sphere in some cases. If $R_1, \ldots, R_{d(S)}$ represent the one-holed tori and four-holed spheres in such a decomposition, then we may view the boundary curves as a subset of a pants decomposition P and perform independent elementary moves on P, changing only the curves in P that do not correspond to boundary components of R_i.

Choosing a bi-infinite geodesic g_i in $\mathbf{P}(R_i)$ (the Farey graph of each R_i), we obtain a natural embedding

$$F \colon \mathbb{Z}^{d(S)} \to \mathbf{P}(S)$$

where the restriction of F to each coordinate axis moves along the geodesic g_i while remaining fixed in each other subsurface R_i.

While it is immediate that the mapping F is 1-Lipschitz, the difficulty in the proof arises in demonstrating that large distances in $\mathbb{Z}^n$ do not contract under F. Work of Masur and Minsky [MM2] guarantees that distance in $\mathbf{P}(S)$ is measured by sums of distances in sufficiently large projections to the curve complexes of subsurfaces in a certain sense. Here, the projection of the image of F onto R_i is precisely the geodesic g_i, so these estimates preclude any worrisome contraction in the image of F.

5 Questions

We close with some questions about the Weil-Petersson metric.

Question 5.1 *What is the geometric rank of the Weil-Petersson metric?*

The geometric rank of X is the dimension of the maximal dimensional quasi-flat in X. An equivalent question is the geometric rank of the pants graph $\mathbf{P}(S)$. The same question can be asked of the mapping class group, or the Teichmüller metric. Answers appear not to be forthcoming, and would be instrumental in considering questions such as the following:

Question 5.2 *Is every self-quasi-isometry of the Weil-Petersson metric a bounded distance from an isometry?*

By a recent theorem of Masur and Wolf [MW2] each isometry of Teichmüller space with the Weil-Petersson metric is induced by an element of the extended mapping class group $\mathrm{Mod}^*(S)$, the group of self-diffeomorphisms of S modulo those isotopic to the identity. This question of quasi-isometric rigidity has been fruitfully understood for higher-rank symmetric spaces (see [EF2] [KL]) via developing a complete understanding of the behavior of quasi-flats.

In particular, one can ask whether the quasi-flats theorem of [EF2] holds in this context:

Question 5.3 *Is every maximal quasi-flat in the completion of the Weil-Petersson metric a bounded distance from a finite number of maximal flats?*

A *flat*, in this context, refers to an *isometric* image of $\mathbb{R}^n$ for some n. While negative curvature precludes the existence of flats in the Weil-Petersson metric, examples of flats in the Weil-Petersson completion appear to arise in the completing locus, where the Teichmüller space decomposes into a product of two smaller Teichmüller spaces after pinching a separating curve on S. One can then imagine that a product of geodesics in such sub-Teichmüller spaces produces a flat in $\mathrm{Teich}(S)$.

It is straightforward to construct examples similar to those of [EF1] where a given quasi-flat does not lie a bounded distance from a single flat of this kind.

Question 5.4 *Are the sectional curvatures of the Weil-Petersson metric bounded away from zero when S is a two-holed torus or five-holed sphere?*

We note that if the sectional curvatures of the Weil-Petersson metric were strictly negative in these cases, this would prove Gromov-hyperbolicity of the Weil-Petersson metrics on Teichmüller spaces for such surfaces. Indications are that sectional curvatures *do* tend to zero as one pinches two disjoint simple closed curves on a surface $X \in \mathrm{Teich}(S)$.

Question 5.5 *Given a point $X \in \mathrm{Teich}(S)$, how does the visual sphere of Weil-Petersson geodesic rays based at X relate to the completion of the Weil-Petersson metric?*

One can imagine that the finite rays based at X are in bijection with points in the completion of the Weil-Petersson metric, and thus such rays can be described by noded Riemann surfaces in the *augmented Teichmüller space* (see [Bers]). Do the infinite rays admit any similarly concise description?

We hope these questions will fuel further interest in properties of the Weil-Petersson metric and its applications to understanding Teichmüller space and the mapping class group.

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