The Weil-Petersson visual sphere

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Abstract

We formulate and describe a visual compactification of the Teichmüller space by Weil-Petersson geodesic rays emanating from a point X. We focus on analogies with Bers's compactification: due to non-completeness, finite rays correspond to cusps, and such cusps are dense in the visual sphere. By analogy with a result of Kerckhoff and Thurston, we show the natural action of the mapping class group does not extend continuously to the visual compactification. We conclude with examples that distinguish the visual boundary from Bers's boundary for Teichmüller space.

1 Introduction

Let $S = S_{g,n}$ be a compact surface of negative Euler-characteristic, of genus g, and with n boundary components. Its *Teichmüller space* $\mathbf{T}(S)$ carries various natural metrics, and for each metric comparisons and analogies are often drawn with the hyperbolic space of the same dimension.

The Weil-Petersson metric, in particular, is a Riemannian metric on $\mathbf{T}(S)$ of variable negative curvature. Scott Wolpert proved that the Weil-Petersson metric, while not complete, is geodesically convex, so its synthetic geometry is tractable.

The hyperbolic space $\mathbb{H}^n$ is compactified by the space of infinite geodesic rays emanating from a given point $x \in \mathbb{H}^n$, its *visual sphere*. Likewise, each point $X \in \mathbf{T}(S)$ has a Weil-Petersson visual sphere, although some geodesic rays emanating from X leave $\mathbf{T}(S)$ in finite time. Such finite rays illustrate the failure of completeness of the Weil-Petersson metric. Nevertheless, geodesic convexity guarantees that the space of such rays compactifies $\mathbf{T}(S)$. We denote this $(6g + 2n - 7)$-dimensional sphere of rays by $\mathcal{V}_X(S)$.

In this paper we give an initial description of this compactification and its interaction with more well-known phenomena in Teichmüller theory, particularly those arising in an analogous compactification due to L. Bers. Here is an example.

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Theorem 1.1. NON-CONTINUITY OF THE MODULAR GROUP ACTION *If* $S = S_{g,n}$ *and* $3g - 3 + n \geq 2$ *then the natural action of the mapping class group* $\mathrm{Mod}(S)$ *on* $\mathbf{T}(S)$ *does not extend continuously to the Weil-Petersson visual sphere* $\mathcal{V}_X(S)$.

The group $\mathrm{Mod}(S)$ of orientation preserving self-homeomorphisms of S up to isotopy, or *mapping class group*, acts by isometries in the Weil-Petersson metric. The theorem is analogous to results of S. Kerckhoff on the visual compactification by *Teichmüller* geodesic rays and to work of Kerckhoff and Thurston on Bers' compactification (see [Ker] and [KT]). In particular, we obtain the same necessary condition for such non-continuity.

Corollary 1.2. *Thurston's compactification for Teichmüller space by the projective measured laminations* $\mathcal{PML}(S) \cong S^{6g+2n-7}$ *is distinct from the compactification of* $\mathbf{T}(S)$ *by the Weil-Petersson visual sphere: the identity map on* $\mathbf{T}(S)$ *does not extend to a homeomorphism from Thurston's boundary to the Weil-Petersson visual sphere.*

The Weil-Petersson visual compactification of Teichmüller space arises from the embedding

$$\mathcal{G}_X \colon \mathbf{T}(S) \to \mathrm{WP}_X(S)$$

of $\mathbf{T}(S)$ into the space $\mathrm{WP}_X(S)$ of Weil-Petersson geodesics joining X to points in $\mathbf{T}(S)$. By geodesic convexity (see [Wol3]) there is a unique Weil-Petersson geodesic $g(X,Y) \in \mathrm{WP}_X(S)$ joining X to Y in $\mathbf{T}(S)$, and the mapping $\mathcal{G}_X$ given by

$$\mathcal{G}_X(Y) = g(X,Y)$$

is then well defined. The negative sectional curvature of the Weil-Petersson metric naturally leads to the following.

Theorem 1.3 (Wolpert). *The embedding* $\mathcal{G}_X$ *is a homeomorphism from* $\mathbf{T}(S)$ *to the space* $\mathrm{WP}_X(S)$ *with the topology of pointwise convergence of parameterizations by the interval proportional to arclength.*

(See [Wol5, Thm. 5]).

Each geodesic $g(X,Y)$, $X \neq Y$ determines a geodesic ray $\vec{g}(X,Y) \in \mathcal{V}_X(S)$ emanating from X and passing through Y. Thus, for any divergent sequence $Y_n \to \infty$ in $\mathbf{T}(S)$ we may extract a convergent subsequence of the rays $\vec{g}(X,Y_n)$ in $\mathcal{V}_X(S)$, so $\mathcal{V}_X(S)$ compactifies $\mathbf{T}(S)$.

The mapping class group $\mathrm{Mod}(S)$ acts on $\mathrm{WP}_X(S)$ by self-homeomorphisms via

$$g(X,Y) \mapsto g(X,\varphi(Y)),$$

where $\varphi \in \mathrm{Mod}(S)$. The natural change of base-point mappings

$$b_{X,Y} \colon \mathrm{WP}_X(S) \to \mathrm{WP}_Y(S)$$

given by

$$b_{X,Y}(\mathcal{G}_X(Z)) = \mathcal{G}_Y(Z)$$

are homeomorphisms for each X and Y.

Our proof of Theorem 1.1 will play on the interaction of these two homeomorphisms. An important step is to describe the interaction of the visual sphere $\mathcal{V}_X(S)$ with the completion $\overline{\mathbf{T}(S)}$ of the Weil-Petersson metric. H. Masur developed the first detailed description of the completion, identifying the completion with the *augmented Teichmüller space* obtained by adjoining *Riemann surfaces with nodes*, singular structures where certain simple closed curves have been pinched to cusps, as limits of pinching degenerations.

To begin with, we show points in the completion $\overline{\mathbf{T}(S)}$ of the Weil-Petersson metric have a natural interpretation in $\mathcal{V}_X(S)$. We call the set $\overline{\mathbf{T}(S)} \setminus \mathbf{T}(S)$ the *frontier* of $\overline{\mathbf{T}(S)}$.

Theorem 1.4. *The set of finite rays in $\mathcal{V}_X(S)$ sit in bijection with the frontier of $\overline{\mathbf{T}(S)}$.*

The theorem sits in analogy to the bijective correspondence between $\overline{\mathbf{T}(S)} \setminus \mathbf{T}(S)$ and the *geometrically finite* locus in the Bers' compactification of Teichmüller space. Indeed, the length of a finite length geodesic based at X that terminates at a noded surface N in the augmented Teichmüller space is approximately the volume of the convex core of the corresponding hyperbolic 3-manifold in the Bers compactification B_X (see [Br3, Thm. 1.2]). Since this volume must then be finite, the corresponding 3-manifold is geometrically finite.

Each finite ray in $\mathcal{V}_X(S)$ determines a simplex σ in the *complex of curves* $\mathcal{C}(S)$: σ is a collection of pairwise disjoint essential simple closed curves on S whose lengths tend to zero on surfaces exiting $\mathbf{T}(S)$ along the ray (we give a more complete description in section 2). As in compactifications of $\mathbf{T}(S)$ due to Thurston and Bers, these "cusps" play the role of rational points, and can be used to encode other points in $\mathcal{V}_X(S)$ via limits.

Theorem 1.5. Cusps are Dense. *The finite rays are dense in the visual sphere $\mathcal{V}_X(S)$.*

Given an infinite ray $\{X_t\}$, in particular, each point along $\{X_t\}$ lies at a uniformly bounded distance from some maximally pinched, or *maximally cusped* Riemann surface lying in the completion of $\mathbf{T}(S)$. Finite rays emanating from X that terminate at these maximally cusped surfaces converge to the ray $\{X_t\}$ in $\mathcal{V}_X(S)$. The analogous motivating result for Bers' compactification was proven by C. McMullen (see [Mc1]).

A key component of our analysis here lies in a description of the limiting behavior of sequences of Weil-Petersson geodesics recently obtained by Wolpert as well as related results of G. Daskalopoulos and R. Wentworth. These investigations were motivated by observations of Sumio Yamada concerning the action of the mapping class group on the Weil-Petersson metric and the behavior of geodesics that encounter the frontier.

In particular, one has the following as a summary of their observations and results:

Theorem 1.6 (Yamada, Daskalopoulos-Wentworth, Wolpert). *The extension of the Weil-Petersson distance function $d_{\mathrm{WP}}(.,.)$ to the distance $d_{\overline{\mathrm{WP}}}(.,.)$ on $\overline{\mathbf{T}(S)}$ gives $\overline{\mathbf{T}(S)}$ a length structure with following properties*

1. *$\overline{\mathbf{T}(S)}$ is a complete geodesic metric space,*
2. *$\overline{\mathbf{T}(S)}$ is* CAT(0), *and*
3. *for each geodesic $\gamma\colon [0,1] \to \overline{\mathbf{T}(S)}$, for which $\gamma(0) \in \mathbf{T}(S)$, one has*
$$\gamma([0,1)) \subset \mathbf{T}(S).$$

Idea of the proof of Theorem 1.1. As in [KT] our argument arises from consideration of the change of base-point map $b_{X,Y}$, and its relevance to the action of the mapping class group on $\mathbf{T}(S)$.

Let $\tau \in \mathrm{Mod}(S)$ be a Dehn-twist about a curve δ. Analyzing families of Weil-Petersson geodesics $g(X, \tau^n(Y))$ joining X to $\tau^n(Y)$, one obtains a sequence of rays in $\mathcal{V}_X(S)$ whose limit (after possibly passing to a subsequence) terminates a node Z in the frontier of the completion $\overline{\mathbf{T}(S)}$.

We show that the point Z depends both on X and on Y. Thus, if X and Y are chosen appropriately, a mapping class φ may be chosen so that $g(\varphi(X), \tau^n(Y))$ limits to a ray in $\mathcal{V}_X(S)$ terminating at $Z' \neq Z$. But a sequence W_n converging to Z directly along the geodesic $g(X, Z)$ has the property that $g(\varphi(X), W_n)$ limits to a ray terminating at Z. This shows that the natural change of base-point map $b_{X,\varphi(X)}$ cannot extend continuously to a map $\mathcal{V}_X(S) \to \mathcal{V}_{\varphi(X)}(S)$. Since the action of φ as a Weil-Petersson isometry on $\mathbf{T}(S)$ sends $g(X, Y)$ to $g(\varphi(X), \varphi(Y))$, if there were a continuous extension of the action $g(X, Y) \mapsto g(X, \varphi^{-1}(Y))$ to $\mathcal{V}_X(S)$, a continuous extension of $b_{X,\varphi(X)}$ to $\mathcal{V}_X(S)$ would follow, giving a contradiction. The theorem follows.

Remark. The key step in the proof of the analogous theorem of Kerckhoff and Thurston for Bers' compactification (see [KT, Thm. 1]) is to show that the *algebraic limit* of the quasi-Fuchsian manifolds $Q(X, \tau^n Y)$ depends simultaneously on the *pair* (X, Y) rather than simply the base-point X. Their argument requires delicate properties of the deformation theory of Kleinian groups. Here, their techniques are recast in a study of the limiting behavior of geodesics in $\overline{\mathbf{T}(S)}$.

The key tool in the proof of Theorem 1.1 is of interest in its own right.

Theorem 1.7. *If $S = S_{g,n}$ and $3g - 3 + n \geq 2$, then the natural homeomorphisms $b_{X,Y}$ of $T(S)$ do not extend to homeomorphisms of the visual spheres $\mathcal{V}_X(S)$ and $\mathcal{V}_Y(S)$.*

Generalizations. While the finite rays in the Weil-Petersson visual sphere correspond to noded Riemann surfaces where finitely many simple closed curves have been pinched to cusps, the infinite rays have no immediate such interpretation.

In joint work with Howard Masur and Yair Minsky, we will show that just as finite rays correspond to geometrically finite points in a Bers compactification, infinite rays in $\mathcal{V}_X(S)$ have a naturally associated geodesic lamination, akin to the *ending laminations* associated to geometrically infinite hyperbolic 3-manifolds. We discuss these laminations and their import in a sequel [BMM].

We close the paper with a result that contrasts the Bers compactification and the Weil-Petersson visual compactification.

Theorem 1.8. *If $S = S_{g,n}$ and $3g - 3 + n \geq 3$, the natural mapping from $\mathrm{WP}_X(S)$ to the Bers slice B_X does not extend to a homeomorphism from $\mathrm{WP}_X(S) \cup \mathcal{V}_X(S)$ to $\overline{B_X}$.*

As with many phenomena in Teichmüller theory, the proof relies on sensitivity to rates of convergence of geometric information on disjoint subsurfaces. Limits in a Bers compactification tend to ignore disparities between the rates of degeneration on disjoint subsurfaces while the Thurston compactification is extremely sensitive to such information (see, e.g., [Br1]). The Weil-Petersson metric is something of a compromise between these different behaviors.

Visual bordifications of CAT(0) spaces. Our notion of a sphere at infinity differs somewhat from some other familiar notions associated to complete CAT(0) spaces (for example, those of [BH, Ch. II.8]). In this context, the *visual bordification* refers to equivalence classes of infinite rays in the complete CAT(0) space X, where two rays are equivalent if they lie within a bounded distance of one another. In our case, the space $\overline{\mathbf{T}(S)}$ is *metrically* complete, if not *geodesically* complete, and thus may be endowed with such a visual bordification. This bordification has the virtue of not being base-point dependent.

As $\overline{\mathbf{T}(S)}$ is not locally compact, however, such a visual bordification of $\overline{\mathbf{T}(S)}$ will not be compact (although isometries of $\overline{\mathbf{T}(S)}$ do extend continuously to such a bordification). Because points in the frontier $\overline{\mathbf{T}(S)} \setminus \mathbf{T}(S)$ have a natural geometric and combinatorial interpretation as noded surfaces, we seek a unified perspective that ties together these points in the completion with the infinite rays by analogy with a Bers compactification.

History and references. The completion of the Weil-Petersson metric had been studied extensively by Wolpert [Wol1, Wol2] and Masur [Mas] in the 1970's. Wolpert proved the geodesic convexity of the Weil-Petersson metric in [Wol3]. More recently, work of S. Yamada sought to give more precise metric descriptions of the behavior of the Weil-Petersson metric near the frontier [Yam]. His study inspired the in independent recent work of G. Daskalopoulos and R. Wentworth [DW] and more recently Wolpert [Wol5].

S. Kerckhoff gave similar analysis of the *Teichmüller boundary* for $\mathbf{T}(S)$ obtained by considering Teichmüller geodesic rays emanating from a point [Ker]. Our work here was inspired in part by the paper [KT]; in each case base-point dependence of the compactification plays a central role.

Acknowledgments. I am grateful to Bill Thurston for suggesting to me possible connections between the Weil-Petersson visual sphere and the closure of a Bers slice. This paper is built largely from the description of the completion of the Weil-Petersson metric presented in [Wol5]. I would like to thank Yair Minsky, Richard Wentworth, and Scott Wolpert for many interesting conversations and suggestions. The initial draft of this paper was written while I was an Assistant Professor at the University of Chicago.

2 Preliminaries

Teichmüller space. The *Teichmüller space* $\mathbf{T}(S)$ parameterizes complete finite area hyperbolic surfaces $X = \mathbb{H}^2/\Gamma$ equipped with homeomorphisms $f\colon \text{int}(S) \to X$ modulo the equivalence relation

$$(f\colon \text{int}(S) \to X) \sim (g\colon \text{int}(S) \to Y)$$

if there is an orientation preserving isometry $\phi\colon X \to Y$ so that $\phi \circ f$ is homotopic to g.

The Teichmüller space has a natural complex structure, and its holomorphic cotangent space $T^*_X\mathbf{T}(S)$ at X is identified with the *quadratic differentials* $Q(X) = \{\varphi(z)dz^2\}$ on X. Let $\rho(z)|dz|$ denote the hyperbolic line-element on X. The *Weil-Petersson metric* is the Hermitian metric on $\mathbf{T}(S)$ arising from the *Petersson scalar product*

$$\langle \varphi, \psi \rangle = \int_X \frac{\varphi\overline{\psi}}{\rho^2} dz d\overline{z}$$

via duality. We will concern ourselves primarily with its Riemannian part g_{WP} and its associated distance function $d_{\text{WP}}(.,.)$ arising from the length-structure induced by g_{WP}.

The Weil-Petersson completion. Let $\mathcal{S}$ denote the collection of all essential isotopy classes of non-peripheral, simple closed curves on S. The *intersection number* $i(\alpha, \beta)$ counts the minimal number of transverse intersection points of representatives of $\alpha, \beta \in \mathcal{S}$.

The *complex of curves* $\mathcal{C}(S)$ is a simplicial complex whose vertices are elements of $\mathcal{S}$ and whose k-simplices span collections $(\alpha_1, \ldots, \alpha_{k+1}) \in \mathcal{S}^{k+1}$ for which $i(\alpha_i, \alpha_j) = 0$ for $1 \leq i < j \leq k$ (see [Har, MM]. A maximal simplex $\sigma \in \mathcal{C}(S)$ determines a *pants decomposition* of S: the vertices σ° of σ determine curves that decompose S into a collection of three-holed spheres, the "pants" of the pants decomposition. .

The *Fenchel-Nielsen coordinates* for $\mathbf{T}(S)$ associated to the vertices σ° of σ measure the *length* and *twisting* of X with respect to σ°:

$$\Pi_{\alpha \in \sigma^\circ}(\ell_\alpha(X), \theta_\alpha(X)) \in \mathbb{R}_+^{|\sigma|} \times \mathbb{R}^{|\sigma|}$$

indicating that X is assembled from hyperbolic pairs of pants with geodesic cuffs of length $\ell_\alpha(X)$ glued together with twisting $\theta_\alpha(X)$ where $\alpha \in \mathcal{S}$ are simple closed curves corresponding to vertices of σ° (see, for example, [IT]). For more mnemonic terminology we will often refer to the "pants decomposition" P as a synonym for the vertices σ° of a maximal simplex in $\mathcal{C}(S)$.

The failure of completeness of the Weil-Petersson metric (see [Wol1]) is remedied by a strong understanding of the points in the completion, developed by H. Masur (see [Mas]). Points in the frontier of the completion $\overline{\mathbf{T}(S)}$ correspond to *noded* hyperbolic surfaces where the curves corresponding to vertices of some (not necessarily maximal) simplex $\sigma \in \mathcal{C}(S)$ have been pinched to cusps. L. Bers described a natural topology in which such noded surfaces can be adjoined at infinity to $\mathbf{T}(S)$ to obtain the *augmented Teichmüller space*, which covers the Mayer-Mumford-Deligne compactification of Moduli space after passing to the quotient of by the mapping class group (see [Brs3]).

One may use *extended Fenchel-Nielsen coordinates* to describe how $\mathbf{T}(S)$ fits together with its metric completion: given a maximal simplex σ, the extended Fenchel Nielsen coordinates

$$\Pi_{\alpha \in \sigma^\circ} \left((\ell_\alpha(X), \theta_\alpha(X)) \in \mathbb{R}_{\geq 0} \times \mathbb{R} / (0, \theta) \sim (0, \theta') \right)$$

allow the parameters $\ell_\alpha(X) = 0$ when X has a node at α but $\theta_\alpha(X)$ is not defined for such points. These coordinates may be used to parameterize possibly noded surfaces in $\overline{\mathbf{T}(S)}$ whose nodes correspond to the vertices of some sub-simplex of σ.

If $\eta \subset \sigma$ is a sub-simplex of σ, the *η-stratum* $\mathbf{T}_\eta$ of $\overline{\mathbf{T}(S)}$ consists of all hyperbolic surfaces with nodes along the vertices of η, and is parameterized in extended Fenchel-Nielsen coordinates with respect to σ by the locus $\{\ell_\alpha(X) = 0 \iff \alpha \in \eta^\circ\}$. We refer the reader to [Brs3], [Ab], [Mas], [Br3], [DW] or [Wol5] for more details.

The extended metric. In [Wol3], Wolpert established that the Weil-Petersson metric is geodesically convex: for each pair $(X, Y) \in \mathbf{T}(S) \times \mathbf{T}(S)$ there is a unique Weil-Petersson geodesic $g(X, Y)$ joining X and Y. Wolpert has recently developed the following picture of the convergence of geodesics $g(X, Y)$ to limiting geodesics in $\overline{\mathbf{T}(S)}$ (see [Wol5]).

Theorem 2.1 (Wolpert). *Let the pairs $(X_n, Y_n) \in \mathbf{T}(S) \times \mathbf{T}(S)$ converge to the pair $(X_\infty, Y_\infty) \in \overline{\mathbf{T}(S)} \times \overline{\mathbf{T}(S)}$, and let $\gamma_n \colon [0,1] \to \mathbf{T}(S)$ be parameterizations proportional to arclength of the geodesics $g(X_n, Y_n)$ with $\gamma_n(0) = X_n$ and $\gamma_n(1) = Y_n$.*

Then the parameterizations γ_n converge pointwise to $\gamma_\infty \colon [0,1] \to \overline{\mathbf{T}(S)}$, the parameterization proportional to arclength of the unique geodesic $g(X_\infty, Y_\infty)$ in from X_∞ to Y_∞.

As a consequence, one has that the completion $\overline{\mathbf{T}(S)}$ is a unique geodesic metric space. Moreover, $\overline{\mathbf{T}(S)}$ is a CAT(0) space. An explicit expansion of the Weil-Petersson metric near the completion reveals that the metric behaves like a product of a Weil-Petersson metric on a lower dimensional Teichmüller space with a cuspidal metric (see [Wol5]). This expansion, together with a re-scaling argument, shows that geodesics that reach the frontier of the completion must terminate at the frontier. Here is one formulation of this result, suggested by Yamada [Yam], and obtained independently in the papers [DW] of Daskalopoulos and Wentworth and [Wol5] of Wolpert.

Theorem 2.2 ([DW, Wol5]). NON-REFRACTION OF GEODESICS. *If the parameterized geodesic* $\gamma\colon [0,1] \to \overline{\mathbf{T}(S)}$ *has* $\gamma(0) \in \mathbf{T}(S)$*, then* $\gamma([0,1))$ *lies entirely within* $\mathbf{T}(S)$*.*

We will apply Theorem 2.2 directly to understand how the frontier of $\overline{\mathbf{T}(S)}$ interacts with the visual sphere. A key consequence is that a geodesic ray $\gamma\colon [0,\infty) \to \overline{\mathbf{T}(S)}$ based at $X \in \mathbf{T}(S)$ has one of two behaviors: either

1. γ *can be continued infinitely:* the image $\gamma([0,\infty)) \subset \mathbf{T}(S)$ is an infinite geodesic in the Weil-Petersson metric, or

2. γ *terminates at* Z *in the frontier:* there is an $s \in [0,\infty)$ so that $\gamma(t)$ converges to a point Z in the frontier of $\overline{\mathbf{T}(S)}$ as $t \to s$, and for any geodesic γ' in $\overline{\mathbf{T}(S)}$ based at X that contains $\gamma[0,s]$ we have $\gamma' = \gamma[0,s]$.

In the latter case, we call γ a *finite ray based at* X and we call Z its *terminal point* in the frontier. These results suffice to prove Theorem 1.4.

Proof of Theorem 1.4. Since each finite ray terminates at some Z in the frontier of $\overline{\mathbf{T}(S)}$ and for each Z in the frontier there is a unique geodesic ray $\vec{g}(X,Z) = g(X,Z)$ terminating at Z (by Theorem 2.2 and the fact that $\overline{\mathbf{T}(S)}$ is CAT(0)), the finite rays are in bijection with points in the frontier of $\overline{\mathbf{T}(S)}$. Thus, Theorem 1.4 is a direct consequence of these results. $\square$

3 Density of cusps

In this section we prove Theorem 1.5.

Theorem 1.5. *The finite rays are dense in the visual sphere* $\mathcal{V}_X(S)$*.*

We begin by recalling the following notions from [Br3] combining work of Bers and Wolpert. Bers proved there is a constant L depending only on S so that given any $X \in \mathbf{T}(S)$ there is a pants decomposition P with

$$\ell_\alpha(X) < L \quad \text{for each } \alpha \in P.$$

Given a pants decomposition P, we denote by $V(P) \subset \mathbf{T}(S)$ the subset

$$V(P) = \{X \in \mathbf{T}(S) \mid \ell_\alpha(X) < L \quad \text{for each } \alpha \in P\}.$$

Then Bers' theorem guarantees that the union over all pants decompositions $\cup_P V(P)$ covers $\mathbf{T}(S)$.

Applying Wolpert's estimates [Wol4, Ex. 4.3] (or, more recently, [Wol5, Cor. 16]) we showed in [Br3, Prop. 2.2] that there is a D depending only on S so that Weil-Petersson diameter satisfies the bound

$$\mathrm{diam}_{\mathrm{WP}}(V(P)) < D.$$

In particular, each point $X \in \mathbf{T}(S)$ lies at a uniformly bounded distance from some maximal node $N(P)$ in the frontier where each curve in P has been pinched to a cusp. (In extended Fenchel Nielsen coordinates about P, the point $N(P)$ is the unique point in $\overline{\mathbf{T}(S)}$ with the coordinates $\{\ell_\alpha(N(P)) = 0 \mid \alpha \in P\}$).

Proof of Theorem 1.5. Let $\gamma\colon [0,\infty) \to \mathbf{T}(S)$ be an infinite geodesic ray in in the visual sphere $\mathcal{V}_X(S)$, parameterized by arclength, where $X = \gamma(0)$. For each $t \in \mathbb{R}$, there is a pants decomposition $P(t)$ so that $\gamma(t)$ lies in $V(P(t))$. By the above discussion, each $Y \in V(P(t))$ is a uniformly bounded distance in the completion $\overline{\mathbf{T}(S)}$ from the maximally noded Riemann surface $N(P(t))$ where all curves in $P(t)$ have been pinched to nodes.

For a given value t_0, Theorem 2.2 guarantees the existence of a Weil-Petersson geodesic $\{\gamma_0(s)\}$, $s \in \mathbb{R}$, parameterized by arclength and terminating at $N(P(t_0))$. Since $N(P(t_0))$ is a uniformly bounded distance from $\gamma(t_0)$, and $\overline{\mathbf{T}(S)}$ is CAT(0), for each $t \le d_{\overline{\mathrm{WP}}}(X, N(P(t)))$ the corresponding points $\gamma(t)$ and $\gamma_0(t)$ are at a uniformly bounded distance.

We let γ_n be the sequence $\{\gamma_n\colon [0,s_n] \to \overline{\mathbf{T}(S)}\}_{n=1}^\infty$ of finite rays in $\mathcal{V}_X(S)$ so that $\gamma_n(s_n) = N(P(n))$. The length of γ_n is growing without bound, and $\gamma_n([0,s_n])$ lies in a uniformly bounded neighborhood of $\gamma([0,n])$ in $\overline{\mathbf{T}(S)}$. Again, since $\overline{\mathbf{T}(S)}$ is CAT(0), it follows that the sequence of rays $\{\gamma_n\}_{n=0}^\infty$ converges to the ray γ in $\mathcal{V}_X(S)$.

Indeed, the angle between the initial directions $\gamma'(0)$ and $\gamma_n'(0)$ of γ and γ_n in the plane they span in $T_X\mathbf{T}(S)$ is at most the angle of the comparison triangle in the Euclidean plane with side-lengths n, s_n, and $d_{\overline{\mathrm{WP}}}(\gamma(n), \gamma_n(s_n))$ which is less than $\mathrm{diam}_{\mathrm{WP}}(V(P(t))) < D$.

Thus, the sequence $\{\gamma_n\}_{n=1}^\infty$ is the desired sequence of finite rays approximating the infinite ray γ. $\square$

4 Iterated twists

Let S be any compact surface of negative Euler characteristic, and let δ be a homotopically essential, non-peripheral, simple closed curve on S.

Let $\mathbf{T}_\delta$ denote the δ-stratum of $\overline{\mathbf{T}(S)}$. In other words, δ is part of a pants decomposition P, so that in extended Fenchel-Nielsen coordinates about P we have

$$\mathbf{T}_\delta = \{X \in \overline{\mathbf{T}(S)} \mid \ell_\delta(X) = 0, \ell_\alpha(X) \neq 0 \text{ for } \alpha \in P \setminus \delta\}$$

Let $\tau_\delta \in \mathrm{Mod}(S)$ be a Dehn-twist about the curve δ. We then prove the following lemma.

Lemma 4.1. *Let X and Y lie in $\mathbf{T}(S)$, and let δ be any essential non-peripheral simple closed curve on S. Then any accumulation point η of the sequence $g(X, \tau_\delta^n Y)$ in $\mathcal{V}_X(S)$ determines a ray that terminates in the stratum $\mathbf{T}_\delta \subset \overline{\mathbf{T}(S)}$.*

Proof. A geodesic γ in $\mathbf{T}(S)$ is said to be ϵ-cobounded if its projection to the moduli space $\mathcal{M}(S)$ lies in the ϵ-thick part, i.e. the region consisting of surfaces whose shortest closed geodesic has length bounded below by ϵ. By Mumford's compactness theorem (see, e.g. [IT, Lem. 6.31]), the ϵ-thick part of $\mathcal{M}(S)$ is compact. Denote by $\mathbf{T}(S)_{\geq \epsilon}$ the preimage of the ϵ-thick part under the natural projection $\mathbf{T}(S) \to \mathrm{Mod}(S)$.

We first claim that for each $\epsilon > 0$ there is an n so that $g(X, \tau_\delta^n Y)$ is not ϵ-cobounded. To see this, note by Theorem 1.1 of [Br3], the Weil-Petersson distance $d_{\mathrm{WP}}(X, \tau_\delta^n Y)$ is uniformly bounded by a constant R_0.

By work of McMullen [Mc3, Sec. 5], for ϵ sufficiently small the Weil-Petersson metric on the ϵ-thick part $\mathbf{T}(S)_{\geq\epsilon}$ is the restriction of a *complete* metric g_ϵ on $\mathbf{T}(S)$ whose sectional curvatures are bounded above and below. Since

$$g_{\mathrm{WP}}|_{\mathbf{T}(S)_{\geq\epsilon}} = g_\epsilon|_{\mathbf{T}(S)_{\geq\epsilon}}$$

if $g(X, \tau_\delta^n Y)$ is ϵ-cobounded then it follows that $g(X, \tau_\delta^n Y)$ are geodesics in the complete metric g_ϵ.

If $\kappa_{\min}$ and $\kappa_{\max}$ are the minimum and the maximum of the sectional curvatures of g_ϵ respectively, there is a constant $V(R_0, \kappa_{\min}, \kappa_{\max})$ so that the ball $B(R_0, X)$ of radius R_0 about any X in the g_ϵ metric on $\mathbf{T}(S)$ has volume bounded by $V(R_0, \kappa_{\min}, \kappa_{\max})$ independently of X. By proper discontinuity of the action of $\mathrm{Mod}(S)$ on $\mathbf{T}(S)$, there is an ϵ_0 so that the ball $B(\epsilon_0, Y)$ satisfies

$$B(\epsilon_0, Y) \cap \tau_\delta^n(B(\epsilon_0, Y)) = \varnothing$$

for all n. Since τ_δ is an isometry for g_{WP}, we have

$$\mathrm{vol}(\tau_\delta^n B(\epsilon_0, Y)) = \mathrm{vol}(B(\epsilon_0, Y))$$

contradicting the uniform volume bound on $B(R_0, X)$ in the g_ϵ-metric. We conclude that there is no ϵ for which $g(X, \tau_\delta^n Y)$ is ϵ-cobounded for each n.

It follows that there are simple closed curves α_n so that the infimum of the length $\ell_{\alpha_n}(Z)$ of α_n over Z lying in the geodesic $g(X, \tau_\delta^n Y)$ tends to zero. We claim that $\alpha_n = \delta$.

By Wolpert's convexity theorem [Wol3], the length $\ell_\alpha(Z)$ is a strictly convex function of Z along the geodesic $g(X, \tau_\delta^n Y)$. We note first that if β is a simple closed curve with geometric intersection number $i(\beta, \delta) = 0$ then

$$\ell_\beta(Y) = \ell_\beta(\tau_\delta^n Y)$$

so, in particular, the length of each such β is bounded along $g(X, \tau_\delta^n Y)$ independent of n. For each such β, there is another β' for which $i(\beta', \delta) = 0$ and $i(\beta, \beta') \neq 0$. By the collar lemma [Bus], it follows that the length of β is also bounded below along $g(X, \tau_\delta^n Y)$ independent of n.

Arguing similarly, the length $\ell_\delta(Z)$ is uniformly bounded above for $Z \in g(X, \tau_\delta^n Y)$ independent of n. It follows that if η is a simple closed curve for which $i(\eta, \delta) \neq 0$, then the length of η is bounded below along $g(X, \tau_\delta^n Y)$ as well. The only remaining possibility is that

$$\inf_{Z \in g(X, \tau_\delta^n Y)} \ell_\delta(Z) \to 0 \quad \text{as} \quad n \to \infty.$$

It follows that the infimum

$$\inf_{Z \in g(X, \tau_\delta^n Y)} d_{\overline{\mathrm{WP}}}(Z, \mathbf{T}_\delta) \to 0 \quad \text{as} \quad n \to \infty$$

as well (see [Wol5, Cor. 16]).

The distance function between corresponding pairs of points along geodesics in $\mathbf{T}(S)$ parameterized proportionally to arclength is strictly convex [Wol5, Prop. 7]. Taking limits, Wolpert shows moreover that the distance between corresponding points along geodesics in $\overline{\mathbf{T}(S)}$ is a convex function [Wol5, Thm. 14]. It follows that there is a unique point $Z_n \in g(X, \tau_\delta^n Y)$ at which $d_{\overline{\mathrm{WP}}}(Z, \mathbf{T}_\delta)$ is minimized along $g(X, \tau_\delta^n Y)$. By the above, the points Z_n eventually lie in every neighborhood of the stratum $\mathbf{T}_\delta$. Moreover, by consideration of the extended Fenchel-Nielsen coordinates on $\mathbf{T}_\delta$ the nearest points $Z_n' \in \mathbf{T}_\delta$ to Z_n lie in a compact subset of $\mathbf{T}_\delta$.

Since $\overline{\mathbf{T}(S)}$ is a unique geodesic space [Wol5, Cor. 6], for any subsequence for which Z_n' converge in $\mathbf{T}_\delta$ to a limit Z_∞', the corresponding geodesics $g(X, Z_n)$ converge to the geodesic in $\overline{\mathbf{T}(S)}$ joining X to Z_∞'. Letting $\gamma\colon [0,1] \to \overline{\mathbf{T}(S)}$ be a parameterization of this limiting geodesic, we have that

$$\gamma([0,1)) \subset \mathbf{T}(S)$$

by Theorem 2.2.

Applying the uniqueness of geodesics in $\overline{\mathbf{T}(S)}$ again, it follows that for any convergent subsequence of $g(X, \tau_\delta^n Y)$ in $\mathcal{V}_X(S)$, the limiting ray is a finite ray joining X to a point $Z_\infty \in \mathbf{T}_\delta$. $\square$

Remark. Recently, Wolpert has given a more complete characterization of the limiting behavior of the geodesic segments $g(X, \tau_\delta^n Y)$ in an effort to analyze the action of $\mathrm{Mod}(S)$ on $\mathbf{T}(S)$ with the Weil-Petersson metric; we refer the reader to [Wol5, Sec. 6].

5 Non-continuity

In this section we prove Theorem 1.1

Theorem 1.1. Non-Continuity of the Modular Group Action *If $S = S_{g,n}$ and $3g-3+n \geq 2$ then the natural action of the mapping class group* $\mathrm{Mod}(S)$ *on* $\mathbf{T}(S)$ *does not extend continuously to the Weil-Petersson visual sphere* $\mathcal{V}_X(S)$.

We employ Lemma 4.1 to establish Theorem 1.7, from which Theorem 1.1 will follow as a corollary.

Theorem 1.7. *If $S = S_{g,n}$ and $3g-3+n \geq 2$, then the natural homeomorphisms $b_{X,Y}$ of $T(S)$ do not extend to homeomorphisms of the visual spheres $\mathcal{V}_X(S)$ and $\mathcal{V}_Y(S)$.*

Proof. If the mapping $b_{X,Y}$ is to extend continuously to $\mathcal{V}_X(S)$, then the image of a finite ray $\eta\colon [0,s] \to \overline{\mathbf{T}(S)}$ under any extension of $b_{X,Y}$ must be the limiting ray for the image path $\{b_{X,Y}(\eta(t))\}_{t=0}^{s}$ in $\mathrm{WP}_Y(S)$. But applying Theorem 2.1, the geodesic segments $\overline{b_{X,Y}(\eta(t))}$ converge to a finite ray $\eta' \in \mathcal{V}_Y(S)$ that terminates at the same point in the frontier of $\overline{\mathbf{T}(S)}$ as the ray η.

Thus, to prove the theorem it suffices to exhibit an X, Y, an $\eta \in \mathcal{V}_X(S)$ and a sequence $X_n \in \mathbf{T}(S)$ so that the sequence of geodesic rays $\vec{g}(X, X_n)$ converges to η but the sequence $\vec{g}(Y, X_n)$ converges to a ray $\eta' \in \mathcal{V}_Y(S)$ that does not have the same terminal point as η.

Lemma 5.1. *Assume the notation of Lemma 4.1. After passing to a subsequence n_j so that $g(X, \tau_\delta^{n_j} Y)$ converges in $\mathcal{V}_X(S)$, the terminal point of the limiting ray for $g(X, \tau_\delta^{n_j} Y)$ in $\mathbf{T}_\delta$ is the same as the terminal point of the limiting ray for $g(Y, \tau_\delta^{-n_j} X)$ in $\mathbf{T}_\delta$.*

Proof. The action of the mapping class τ_δ on $\mathbf{T}(S)$ is isometric, and extends to the identity on the stratum $\mathbf{T}_\delta$ (consider the action of τ_δ on the extended Fenchel-Nielsen coordinates at $\mathbf{T}_\delta$).

Let Z_n be the closest point to $\mathbf{T}_\delta$ along $g(X, \tau_\delta^n Y)$. Then the image $\tau_\delta^{-n}(g(X, \tau_\delta^n Y))$ of the geodesic $g(X, \tau_\delta^n Y)$ under the isometry τ_δ^{-n} of $\overline{\mathbf{T}(S)}$ comes closest to $\mathbf{T}_\delta$ at the point $\tau_\delta^{-n}(Z_n)$. The geodesic γ in $\overline{\mathbf{T}(S)}$ joining Z_n to the nearest point Z_n' in $\mathbf{T}_\delta$, has image $\tau_\delta^{-n}(\gamma)$ a geodesic joining $\tau_\delta^{-n}(Z_n)$ to the same point Z_n'.

Since we have

$$d_{\overline{\mathrm{WP}}}(Z_n, Z_n') = d_{\overline{\mathrm{WP}}}(\tau_\delta^{-n}(Z_n), Z_n')$$

the points Z_n and $\tau_\delta^{-n}(Z_n)$ converge to the (same) limit Z_∞' of Z_n' after passing to a subsequence.

Applying Theorem 2.2, Z_∞' is the unique accumulation point of the geodesics $g(X, \tau_\delta^{n_j}(Y))$ and likewise of $g(Y, \tau_\delta^{-n_j}(X))$. For such a subsequence, then, the limiting ray for $\{g(X, \tau_\delta^{n_j}(Y))\}$ in $\mathcal{V}_X(S)$ joins X to $Z_\infty' \in \overline{\mathbf{T}(S)}$ and the limiting ray for $\{g(Y, \tau_\delta^{-n_j}(X))\}$ in $\mathcal{V}_Y(S)$ joins Y to $Z_\infty' \in \overline{\mathbf{T}(S)}$. $\square$

Continuation of the proof of Theorem 1.7. Let P denote a pants decomposition of S containing a curve δ so that either

1. one component of $S \setminus \delta$ has closure a three-holed sphere W, or
2. δ is non-separating.

Let $\{(\ell_\alpha(X), \theta_\alpha(X)) \mid \alpha \in P\}$ be extended Fenchel-Nielsen coordinates for $X \in \mathbf{T}(S) \cup \mathbf{T}_\delta$ with respect to P (where $\mathbf{T}_\delta$ represents the locus $\{\ell_\delta = 0\}$ as in section 2).

Let T denote either

1. the complement $T = \overline{S \setminus W}$ of W in S in case (1), or
2. the complementary subsurface $T = \overline{S \setminus \mathcal{N}(\delta)}$ of an annular collar $\mathcal{N}(\delta)$ about δ in case (2).

In either case, since we have assumed $S = S_{g,n}$ where $3g - 3 + n \geq 2$, $\mathbf{T}(T)$ has positive dimension, and the stratum $\mathbf{T}_\delta$ is isomorphic to $\mathbf{T}(T)$. The distance induced on $\mathbf{T}_\delta$ by $\overline{\mathbf{T}(S)}$ agrees with Weil-Petersson distance on $\mathbf{T}(T)$ between pieces of the corresponding noded surfaces that are marked by T. Let $g(X_0, Y_0)$ be a geodesic segment of length 1 in $\mathbf{T}_\delta$.

Let $\epsilon > 0$. In our extended Fenchel-Nielsen coordinates near $\mathbf{T}_\delta$, we let X_ϵ and Y_ϵ be determined by

$$(\ell_\alpha(X_\epsilon), \theta_\alpha(X_\epsilon)) = (\ell_\alpha(X_0), \theta_\alpha(X_0)) \quad \text{and} \quad (\ell_\alpha(Y_\epsilon), \theta_\alpha(Y_\epsilon)) = (\ell_\alpha(Y_0), \theta_\alpha(Y_0))$$

for $\alpha \in P \setminus \delta$, and

$$(\ell_\delta(X_\epsilon), \theta_\delta(X_\epsilon)) = (\epsilon, 0) \quad \text{and} \quad (\ell_\delta(Y_\epsilon), \theta_\delta(Y_\epsilon)) = (\epsilon, 0).$$

Consider the sequence of geodesics segments

$$\{g(X_\epsilon, \tau_\delta^n(Y_\epsilon))\} \;\text{ in }\; \mathrm{WP}_{X_\epsilon}(S).$$

Let $Z_\epsilon \in \mathbf{T}_\delta$ be the terminal point of the limiting ray for $g(X_\epsilon, \tau_\delta^n(Y_\epsilon))$ after passing to a subsequence n_j. As $\epsilon \to 0$, the distance $d_{\overline{\mathrm{WP}}}(X_\epsilon, X_0)$ tends to 0 and likewise $d_{\overline{\mathrm{WP}}}(Y_\epsilon, Y_0)$ tends to 0. By Theorem 2.1, the geodesics $g(X_\epsilon, Y_\epsilon)$ admit parameterizations by $[0, 1]$ proportional to arclength that converge pointwise to a parameterization of the geodesic $g(X_0, Y_0)$ proportional to arclength as ϵ tends to zero. Since $\overline{\mathbf{T}(S)}$ is CAT(0), it follows that there are numbers $\epsilon_i \to 0$ so that Z_{ϵ_i} converges to a point on the geodesic $g(X_0, Y_0)$.

Let Z_0 be this limit. Assume for the moment that Z_0 lies in the interior of $g(X_0, Y_0)$. Then choosing a new surface $X_0' \neq X_0$ at distance 1 from Y_0 in $\mathbf{T}_\delta$ produces a point $Z_0' \in g(X_0', Y_0)$ by the same iterative procedure, after passing to perhaps further subsequences.

Here, Z_0' is a limit of terminal points Z'_{ϵ_j} of the limiting rays for the sequences $g(X'_{\epsilon_j}, \tau^n_\delta(Y_{\epsilon_j}))$ for a subsequence of values $\epsilon_j \in \{\epsilon_i\}$ tending to zero.

Since $g(X_0, Y_0)$ and $g(X_0', Y_0)$ do not meet on their interior, we may take ϵ sufficiently small so that Z_ϵ and Z'_ϵ are distinct points in $\mathbf{T}_\delta$. But Z_ϵ is the terminal point of the limiting ray for $g(X_\epsilon, \tau^n_\delta(Y_\epsilon))$ while Z'_ϵ is the terminal point of the limiting ray for $g(X'_\epsilon, \tau^n_\delta(Y_\epsilon))$. Hence, the terminal points for the limiting rays depend on the choice of base-point, provided Z_0 lies in the interior of $g(X_0, Y_0)$.

If $Z_0 = X_0$, then the same conclusion holds, since X_0 does not lie on $g(X_0', Y_0)$. If, however, $Z_0 = Y_0$, we apply Lemma 5.1 to conclude that Z_ϵ is also the terminal point for the limiting ray for $g(Y_\epsilon, \tau^n_\delta(X_\epsilon))$ and argue symmetrically by reversing the roles if X_ϵ and Y_ϵ, perturbing the point Y_0 to obtain the contradiction. This proves the theorem. $\square$

Having proven Theorem 1.7, it remains to deduce Theorem 1.1 as a consequence.

Proof of Theorem 1.1. Choose X_0 and Y_0 as in the proof of Theorem 1.7 with the added condition that for some mapping class $\varphi \in \mathrm{Mod}(S)$ that fixes δ up to isotopy, we have $\varphi(X_0) \neq X_0$, $\varphi(Y_0) \neq Y_0$ and $\varphi^{\pm 1}(Y_0) \neq X_0$ (it is an easy exercise to find such a triple (X_0, Y_0, φ)).

Then if the action of φ is to extend continuously to $\mathcal{V}_X(S)$, the limit η of the convergent sequence $g(X, Y_n)$ must map to the limit $\varphi(\eta)$ of convergent sequence $g(X, \varphi(Y_n))$, and this limit $\varphi(\eta)$ must not depend on the approach to η. In particular, if Y_n' is another sequence for which $g(X, Y_n')$ converges to η, then $g(X, \varphi(Y_n'))$ must converge to $\varphi(\eta)$ if φ is to extend continuously to $\mathcal{V}_X(S)$.

Since φ acts isometrically on the completion $\overline{\mathbf{T}(S)}$, the sequence $g(X, \varphi(Y_n))$ converges to a finite ray in $\mathcal{V}_X(S)$ that terminates at Z if and only if the geodesics $g(\varphi^{-1}X, Y_n)$ converge to a ray that terminates at $\varphi^{-1}(Z)$. This reduces the continuity of the action of φ to the continuity of the change of base-point $X \mapsto \varphi^{-1}(X)$. Thus, provided φ satisfies the above conditions, the argument for Theorem 1.7 may be applied to points $X = X_\epsilon$ and $Y = Y_\epsilon$ in $\mathbf{T}(S)$ in close neighborhoods of X_0 and Y_0 (as above) to show that the action of φ cannot extend continuously to the visual compactification $\mathcal{V}_X(S)$. $\square$

6 The Bers compactification and the visual compactification

Having illustrated many similarities of the Weil-Petersson visual sphere with the Bers compactification, we conclude the paper with an illustration that Bers compactification carries different asymptotic data about divergent surfaces in $\mathbf{T}(S)$ than the visual sphere.

Theorem 1.8. *If $S = S_{g,n}$ and $3g - 3 + n \geq 3$, the natural mapping from $\mathrm{WP}_X(S)$ to the Bers slice B_X does not extend to a homeomorphism from $\mathrm{WP}_X(S) \cup \mathcal{V}_X(S)$ to $\overline{B_X}$.*

Given a pair $(X, Y) \in \mathbf{T}(S) \times \mathbf{T}(S)$ Bers proved there is a unique *simultaneous uniformization* of X and Y, namely, a *quasi-Fuchsian* hyperbolic 3-manifold $Q(X, Y)$ homeomorphic to $S \times \mathbb{R}$ with X and Y as its conformal *ideal boundary.* Here, $Q(X, Y) = \mathbb{H}^3/\Gamma(X, Y)$ where $\Gamma(X, Y)$ is a discrete subgroup of $\mathrm{PSL}_2(\mathbb{C}) = \mathrm{Isom}^+\mathbb{H}^3$, the orientation preserving isometry group of $\mathbb{H}^3$. The group $\Gamma(X, Y)$ simultaneously uniformizes X and Y as the quotient of its *domain of discontinuity* where $\Gamma(X, Y)$ acts properly discontinuously.

Bers showed, moreover, that this product of Teichmüller spaces parameterizes the quasi-Fuchsian manifolds within the *algebraic deformation space* $AH(S)$ of hyperbolic 3-manifolds homeomorphic to $\mathrm{int}(S) \times \mathbb{R}$ where ∂S corresponds to cusps. The *Bers slice* B_X is obtained by considering the slice $\{X\} \times \mathbf{T}(S)$ in the product $\mathbf{T}(S) \times \mathbf{T}(S)$.

Bers proved [Brs2] the slice B_X describes a precompact subset of the algebraic deformation space. Points in the boundary ∂B_X represent of certain types of degenerations of the of the 3-dimensional hyperbolic geometry. Each point $M \in \partial B_X$ is a limit of a sequence $\{Q(X, Y_n)\}$ and the geometry of the limit reflects the degeneration of the surfaces Y_n. The surface X persists as a component of the conformal boundary of M.

In particular, given an $M \in \partial B_X$, either

1. M is a *cusp*: there is a simplex $\eta \in \mathcal{C}(S)$ so that the curves in η° have been pinched to cusps, or
2. M is *totally degenerate*: the conformal boundary of M consists only of the surface X.

(Note that these two cases are not mutually exclusive).

If the total area of the conformal boundary ∂M (in the Poincaré metrics on the components) is equal to $4\pi|\chi(S)|$, then M represents a *geometrically finite* point in ∂B_X. We may then view $\partial M \setminus X$ as a noded Riemann surface Z, pinched along the parabolics of M. Geometrically finite points in the Bers compactification boundary are precisely the augmented Teichmüller space [Ab], which is also precisely the completion $\overline{\mathbf{T}(S)}$ [Mas]. Thus by Theorem 1.4, the finite rays in the visual sphere $\mathcal{V}_X(S)$ are identified with the geometrically finite points in the Bers boundary ∂B_X.

For more on Bers' simultaneous uniformization theorem and the resulting compactification, we refer the reader to [Brs1], [Brs2], [Ab] [Mc2], or [Br2].

We now prove Theorem 1.8.

Proof. Consider two surfaces $Y \neq Y'$ in $\mathbf{T}(S)$, and let $d_{\mathrm{WP}}(Y, Y') = d_0$. Then for any mapping class $\psi \in \mathrm{Mod}(S)$ with positive translation distance in the Weil-Petersson metric, we have

$$d_{\mathrm{WP}}(\psi^n(Y), \psi^n(Y')) = d_0 \quad \text{and} \quad d_{\mathrm{WP}}(X, \psi^n(Y)) \to \infty$$

as $n \to \infty$. Note that in particular, if the rays $\vec{g}(X, \psi^n(Y))$ converge in $\mathcal{V}_X(S)$ then the rays $\vec{g}(X, \psi^n(Y'))$ also converge in $\mathcal{V}_X(S)$ to the same limit.

Let $T \subset S$ be an essential subsurface that is either a four-holed sphere or a one-holed torus, and so that the complement $R = \overline{S \setminus T}$ is connected. Let $\varphi \in \mathrm{Mod}(S)$ be a mapping class with a representative φ^* in the orientation preserving self-homeomorphisms of S with the following properties:

1. φ^* preserves T up to isotopy,
2. the restriction of φ^* to T determines a pseudo-Anosov element ψ in $\mathrm{Mod}(T)$, and
3. the restriction of φ^* to $S \setminus T$ is isotopic to the identity.

Then φ has positive Weil-Petersson translation distance on $\mathbf{T}(S)$.

Let η be the simplex in $\mathcal{C}(S)$ whose vertices correspond to the boundary of T, and let $\mathbf{T}_\eta$ denote the associated stratum of $\overline{\mathbf{T}(S)}$ where all of the curves corresponding to η° are pinched. Then $\mathbf{T}_\eta$ has the structure

$$\mathbf{T}_\eta = \mathbf{T}(T) \times \mathbf{T}(R).$$

Choose noded surfaces Y_n and Y_n' in $\mathbf{T}_\eta$ as follows. Let $W \in \mathbf{T}(T)$ and Z and Z' be distinct surfaces in $\mathbf{T}(R)$. Then we let

$$Y_n = (\psi^n(W), Z) \quad \text{and} \quad Y_n' = (\psi^n(W), Z').$$

The induced metric on the stratum $\mathbf{T}_\eta$ is the product of the Weil-Petersson metrics from the Teichmüller spaces $\mathbf{T}(T)$ and $\mathbf{T}(R)$, so the distance

$$d_{\overline{\mathrm{WP}}}(Y_n, Y_n') = d_{\mathrm{WP}}(Z, Z')$$

which is, in particular, independent of n.

It follows that there are integers n_j so that the rays $\vec{g}(X, Y_{n_j})$ and $\vec{g}(X, Y_{n_j}')$ in $\mathcal{V}_X$ have the same limit. We claim that the corresponding limits in the Bers compactification $\overline{B_X}$ are distinct.

Let $\overline{Q}_n = \overline{Q}(X, Y_n) \in \partial B_X$ denote the cusp in the Bers' boundary where η° represents the accidental parabolics for $\overline{Q}(X, Y_{n_j})$, and the conformal boundary $\partial\overline{Q}(X, Y_{n_j})$ consists of X, $\psi^n(W)$, and Z. Let $\overline{Q}' = \overline{Q}(X, Y_n')$ be determined likewise with Z' in place of Z.

By results of [Br2], these sequences $\overline{Q}_n$ and $\overline{Q}_n'$ converge to limits Q_∞ and Q_∞' in the Bers boundary ∂B_X. The surface Z lies in the conformal boundary of Q_∞ while the surface Z' lies in the conformal boundary of Q_∞'. Via the markings of Q_∞ and Q_∞', the surfaces Z and Z' correspond to conjugate subgroups of $\pi_1(S)$ under the natural inclusion maps $\pi_1(Z) \hookrightarrow \pi_1(Q_\infty)$ and $\pi_1(Z') \hookrightarrow \pi_1(Q_\infty')$. If these limits Q_∞ and Q_∞' were the same, we would conclude that $Z = Z'$ contradicting our assumptions. $\square$

Remark: Using iteration of commuting pseudo-Anosov mapping classes ψ_1 and ψ_2 supported on disjoint subsurfaces of S, one can also exhibit examples of sequences

$$Y_n = \psi_1^{a_n} \circ \psi_2^{b_n}(Y)$$

that converge in Bers boundary whose corresponding rays do not converge in the Weil-Petersson visual sphere $\mathcal{V}_X(S)$. By letting a_n tend to ∞ much more quickly than b_n, say, one may force the effect of ψ_1 to dominate the direction of the ray $\vec{g}(X, Y_n)$, forcing a different limiting direction from that obtained by taking $a_n = b_n$. Nevertheless, by the arguments of [Br2] such sequences may be chosen to converge in the Bers boundary.

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